Solution: Exercise 2 (Part 1)

1. Determine the truth value of each of these statements. Explain your answer.

(a) $\forall n \in \mathbb{Z}, \quad n-1<n$ (b) $\forall n \in \mathbb{Z}, \quad n \leq 10n$
(c) $\exists n \in \mathbb{Z}, \quad 2n=-n$ (d) $\exists n \in \mathbb{Z}^-, \quad n=\frac{1}{n}$

Answer:

(a) $\forall n \in \mathbb{Z}, \quad n-1<n$
This statement is **true**, because if we subtract both sides of inequality by n: for any $n \in \mathbb{Z}$:

$$n-1<n \qquad \Rightarrow \qquad -1<0,$$

which is true for all n. Note: other simple argument can also be used here. ■

(b) $\forall n \in \mathbb{Z}, \quad n \leq 10n$
This statement is **false**, because if let $n<0$, then $n \leq 10n$ is not true. E.g.a counterexample is $n=-1$, we have $-1<-10$, which is false. ■

(c) $\exists n \in \mathbb{Z}, \quad 2n=-n$
This statement is **true**, because we can find at least one value of $n \in \mathbb{Z}$, i.e. $n=0$, such that $2n=-n=0$. ■

(d) $\exists n \in \mathbb{Z}^-, \quad n=\frac{1}{n}$
This statement is **true**, because we can find at least one value of $n \in \mathbb{Z}^-$, i.e. $n=-1 \in \mathbb{Z}^-$, such that $n=\frac{1}{n}$ or $-1=\frac{1}{-1}$ is true. ■

2. Let $\mathbb{R}$ be the domain of x. Determine the **truth set** for each of these statements.

(a) $P(x)$: "$x+1<2x$" (b) $P(x)$: "$x^2<4$ and $x \leq 0$"

Answer:
(a) $P(x)$: "$x+1<2x$"
$P(x)$ is true when

$$x+1<2x \qquad \text{or} \qquad x+1-x<2x-x \qquad \text{or} \qquad 1<x.$$

That is, the truth set is $\{x \in \mathbb{R} | x>1\}=(1,\infty)$. ■

(b) $P(x)$: "$x^2<4$ and $x \leq 0$"
$P(x)$ is true when $x^2<4$ and $x \leq 0$ are true. Notice that

$$x^2<4 \qquad \equiv \qquad x^2-4<0 \qquad \equiv \qquad (x-2)(x+2)<0 \qquad \equiv \qquad -2<x<2.$$

That is, we want $-2<x<2$ and $x \leq 0$ to be true at the same time, which occurs when $-2<x \leq 0$ So the truth set is $\{x \in \mathbb{R} | -2<x \leq 0\}=(-2,0]$. ■

3. Let $Q(x, y)$ be the statement "$x + y = x - y$." If the domain for both variables consists of all integers, determine the truth values of the following statements. Explain you answer.

   (a) $Q(1, 1)$ (b) $Q(2, 0)$ (c) $\forall y, Q(1, y)$

   (d) $\exists x, Q(x, 2)$ (e) $\forall x \exists y, Q(x, y)$ (f) $\forall y \exists x, Q(x, y)$

   Answer:

   (a) $Q(1, 1)$:"$1 + 1 = 1 - 1$" or "$2 = 0$" is false.
   (b) $Q(2, 0)$:"$2 + 0 = 2 - 0$" or "$2 = 2$" is true.
   (c) $\forall y, Q(1, y)$:"$1 + y = 1 - y$" or "$y = -y$" is false (a counterexample is $y = 1$).
   (d) $\exists x, Q(x, 2)$:"$x + 2 = x - 2$" or "$-2 = 2$" is always false for any x.
   (e) $\forall x \exists y, Q(x, y)$ is true because for any fixed x, we can choose $y = 0$, so that "$x + 0 = x - 0$" which is true.
   (f) $\forall y \exists x, Q(x, y)$ is false because when we use a fixed value of y that is nonzero, we have "$x + y = x - y$" which will implies "$y = -y$" and this is false. A counterexample is when $y = 1$, we have "$x + 1 = x - 1$" which will implies "$1 = -1$" and we cannot find any x that make this statement true.

4. Let $Q(x, y, z)$ be the statement "$x + y = z$." Let the domain of all variables be the set of all real numbers. Determine the truth value of the statement $\exists z \forall x \forall y,\ Q(x, y, z)$. Explain your answer.

   Answer:

   This statement is false because there is a real number z such that"$x + y = z$" is not true. In particular, for each fixed value of z, then we can always find x and y (e.g. $x = z$, $y = 1$) such that $x + y \neq z$. E.g.
   $z = 1$, set $x = 1$, $y = 1$ we will see that the statement is not true;
   $z = 2$, set $x = 2$, $y = 1$ we will see that the statement is not true;
   $z = 3$, set $x = 3$, $y = 1$ we will see that the statement is not true....etc.
   Note: other counterexamples could be used too.

5. Let $P(x)$, $Q(x)$, $R(x)$, and $S(x)$ be the statements "x is a hummingbird," "x is large," "x lives on honey," and "x is colorful," respectively. Assuming that the domain consists of all birds, express the statements in the argument using quantifiers and $P(x)$, $Q(x)$, $R(x)$, and $S(x)$ and determine whether this argument is valid or invalid by applying the equivalences of statement forms and the inference rule(s).

   "All hummingbirds are colorful."
   "No large birds live on honey."
   "Birds that do not live on honey are not colorful."
   ∴ "Hummingbirds are not large birds."

   Answer:

   Let D be the set of all birds.
   (i)"All hummingbirds are colorful." $\equiv \forall x \in D, P(x) \to S(x)$
   (ii)"No large birds live on honey."
   $\equiv \forall x \in D, Q(x) \to \sim R(x) \equiv \forall x \in D, R(x) \to \sim Q(x)$ (contrapositive)

(iii) "Birds that do not live on honey are not colorful."
$\equiv \forall x \in D, \sim R(x) \rightarrow \sim S(x) \equiv \forall x \in D, S(x) \rightarrow R(x)$ (contrapositive)
(iv) $\therefore$ "Hummingbirds are not large birds." $\equiv \forall x \in D, P(x) \rightarrow \sim Q(x)$
So we have from (i) $\rightarrow$ (iii) $\rightarrow$ (ii), we will get the conclusion (iv): $\forall x \in D$,

$$(i)\ P(x) \rightarrow S(x) \quad , \quad (iii)\ S(x) \rightarrow R(x) \quad , \quad (ii)\ R(x) \rightarrow \sim Q(x)\ , \quad (iv)\ P(x) \rightarrow \sim Q(x),$$

and by using rule of inference **universal transitivity**, this argument is valid.

6. Let $\mathbb{Z}^+$ be the domain of x. Determine whether the following statements are true or false. Give a counterexample for each false statement.
(a) $\sqrt{x} > 1 \Rightarrow x^2 > 23$, (b) $\sqrt{x} > 2 \Leftrightarrow x^2 > 23$.

Answer:

-Notice that $\sqrt{x} > 2$ only when $x > 4$.
So " $\sqrt{x} > 2$ " is true when $x \in \{5, 6, 7, 8, \dots\}$.
Hence, truth set for predicate: " $\sqrt{x} > 2$ " is $\{5, 6, 7, 8 \dots, \}$ or $\mathbb{Z}^+/\{1, 2, 3, 4\}$.
-Notice that $x^2 > 23$ only when $x > \sqrt{23}$ for $x \in \mathbb{Z}^+$.
So "$x^2 > 23$" is true when $x \in \{5, 6, 7, 8, \dots\}$.
Hence, truth set for predicate: "$x^2 > 23$" is $\{5, 6, 7, 8 \dots, \}$ or $\mathbb{Z}^+/\{1, 2, 3, 4\}$.
(a) $\sqrt{x} > 1 \Rightarrow x^2 > 23$ is false since the truth set of "$x^2 > 23$" is not contained in the truth set of "$\sqrt{x} > 1$." In particular,

"$\sqrt{x} > 1$" is true when $x \in \{2, 3, 4, 5 \dots\}$ and hence its truth set is $T_1 := \{2, 3, 4, 5 \dots\}$.
"$x^2 > 23$" is true when $x \in \{5, 6, 7, 8 \dots, \}$ and hence its truth set is $T_2 := \{5, 6, 7, 8 \dots, \}$.

Since $T_1 \nsubseteq T_2$, this statement is false. Also, a counterexample is $x = 2$ or any integer $x \geq 2$. (Note: it is enough the just give a counterexample here).
(b) $\sqrt{x} > 2 \Leftrightarrow x^2 > 23$ is true because, from above, the truth sets of " $\sqrt{x} > 2$ " and "$x^2 > 23$" are the same.

7. Write a negation for each statement without using *the negation symbol* "$\sim$."

(a) $\exists x \in \mathbb{R}$, $(x - 2)(x + 1) > 0$ if and only if $x > 2$ or $x < -1$.
(b) $\forall \varepsilon \exists \delta \forall x \, (|x - a| < \delta \rightarrow |f(x) - L| < \varepsilon)$

Answer:

(a) Let $P(x)$ be $(x - 2)(x + 1) > 0$, $Q(x)$ be $x > 2$, and $R(x)$ be $x < -1$. Note that

$$[P(x) \leftrightarrow (Q(x) \vee R(x))] \equiv [P(x) \rightarrow (Q(x) \vee R(x))] \wedge [(Q(x) \vee R(x)) \rightarrow P(x)]$$

$$\sim [P(x) \leftrightarrow (Q(x) \vee R(x))] \equiv \sim [P(x) \rightarrow (Q(x) \vee R(x))] \vee \sim [(Q(x) \vee R(x)) \rightarrow P(x)]$$

and since we used the fact that

$$\sim [P(x) \rightarrow (Q(x) \vee R(x))] \equiv P(x) \wedge \sim (Q(x) \vee R(x)) \equiv P(x) \wedge (\sim Q(x) \wedge \sim R(x))$$

and

$$\sim [(Q(x) \vee R(x)) \rightarrow P(x)] \equiv (Q(x) \vee R(x)) \wedge \sim P(x)$$

then the negation of the given statement is

$$\exists x \in \mathbb{R}, [P(x) \wedge (\sim Q(x) \wedge \sim R(x))] \vee [(Q(x) \vee R(x)) \wedge \sim P(x)]$$

or

$\exists x \in \mathbb{R}, [((x-2)(x+1) > 0) \wedge (x \leq 2 \wedge x \geq -1)] \vee [(x > 2 \vee x < -1) \wedge (x-2)(x+1) \leq 0]$.

(b) $\sim \forall \varepsilon \exists \delta \forall x\, (|x-a| < \delta \rightarrow |f(x) - L| < \varepsilon)$
$\equiv \exists \varepsilon \sim [\exists \delta \forall x\, (|x-a| < \delta \rightarrow |f(x) - L| < \varepsilon)]$
$\equiv \exists \varepsilon \forall \delta \sim [\forall x\, (|x-a| < \delta \rightarrow |f(x) - L| < \varepsilon)]$
$\equiv \exists \varepsilon \forall \delta \exists x \sim (|x-a| < \delta \rightarrow |f(x) - L| < \varepsilon)$
$\equiv \exists \varepsilon \forall \delta \exists x\, (|x-a| < \delta \ \wedge \ |f(x) - L| \geq \varepsilon)$

8. Show that each of the following arguments is valid by **universal modus ponens** , **universal modus tollens** or **universal transitivity**, or show that it is invalid from the **converse error** or the **inverse error**. In addition, use also the **diagram** to confirm that each argument is valid or invalid.

(a) All rabbits like vegetable. My pet is not a rabbit. Therefore, my pet does not like vegetable.
(b) Everyone who eats fruit every day is healthy. Linda is not healthy. Therefore, Linda does not eat fruit every day.

Answer:

(a) We can transform the given argument in the quantified form of **inverse error:**

$$\forall x, P(x) \rightarrow Q(x)$$
$$\sim P(a) \text{ for a particular } a = \text{ my pet}$$
$$\therefore \sim Q(a)$$

where $P(x)$ is defined as "x is a rabbits" and $Q(x)$ is defined as "x likes vegetable." Hence the argument is **invalid**.

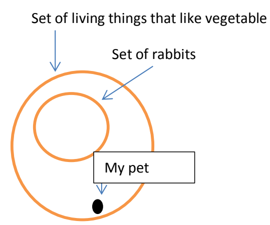

Figure 1: Problem 8 (a)

From the diagram, since it is possible that the "my pet " is still inside the set of living things that like vegetable, even if "my pet " is not in the set of rabbits, then the statement is **invalid**.

(b) First the given premise " Everyone who eats fruit every day is healthy" can be written in terms of *if-then statement* as "$\forall x$, if x eats fruits everyday, then x is healthy."

Then, the given argument can be re-written in a valid form of **universal modus tollens**:

$$\forall x, P(x) \rightarrow Q(x)$$
$$\sim Q(a) \text{ for a particular } a = \text{Linda}$$
$$\therefore \sim P(a),$$

where $P(x)$ is defined as "x eats fruit everyday" and $Q(x)$ is defined as "x is healthy." Hence the argument is **valid** by **universal modus tollens**.

Diagram

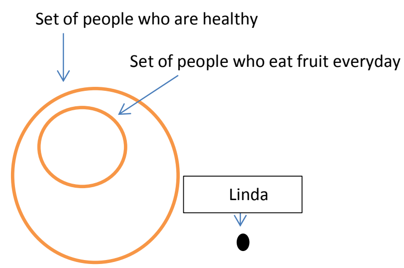

Figure 2: Problem 8 (b)

From the diagram, since Linda is not in the set of people who are healthy, which contains the set of people who eats fruit everyday. So it is impossible that Linda is in the set of people who eats fruit everyday. Hence the conclusion that Linda does not eat fruit everyday is **valid**.