

CHAPTER 6

Optimization without Constraints: One Independent Variable Case

Topics:

- ➔ Maximize profits
 - Competitive market case
 - Monopoly case
- ➔ Effects of taxes
 - Lump-sum tax
 - Profit tax
 - Excise tax
- ➔ Maximization of tax revenue

Our attention will be turned to the study of goal equilibrium, in which the equilibrium state is defined as the optimum position for a given economic unit(e.g. a household, a business firm) and in which the said economic unit will be deliberately striving for attainment of that equilibrium.

We will use the optimization to study how consumers, producers, or public policy makers choose their best alternatives given their limited resources, as the essence of the optimization problem is to choose the BEST alternative available.

Utility Maximization:

$$\max_x U(x)$$

Profit Maximization

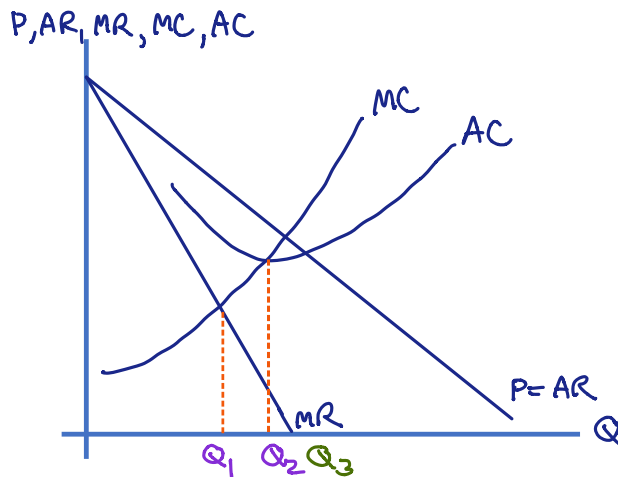
$$\max_Q \pi(Q) \quad \rightarrow MR = MC \rightarrow Q_1$$

Cost Minimization

$$\min_Q C(Q), \min_Q AC(Q) \rightarrow \frac{dAC}{dQ} = 0 \rightarrow Q_2$$

Total Revenue Maximization

$$\max_Q TR(Q) \quad \rightarrow MR = 0 \rightarrow Q_3$$



Tax Revenue Maximization

$$\max_t \text{Tax Revenue}$$

Social Welfare Maximization (later)

$$\max_{x_1, \dots, x_n} \text{Social Welfare Function}$$

Example: $TC = f(Q) = Q^3 - 4Q^2 + 8Q$, Find Q that minimizes AC

$$AC(Q) = Q^2 - 4Q + 8$$

$$\min_Q AC(Q)$$

$$\text{FONC: } \frac{dAC}{dQ} = 2Q - 4 = 0$$

$$Q^* = 2 \quad \#$$

$$\text{SOSC: } \frac{d^2AC}{dQ^2} = 2 > 0$$

$$\therefore AC \text{ at } Q^* \text{ is } AC_{\min}$$

$$AC_{\min} = 2^2 - 4(2) + 8 = 4 \quad \#$$

H.W. Find critical values and the maximum/minimum of the following functions:

$$\text{✂ } y = f(x) = x^3 - 12x^2 + 36x + 8$$

$$\text{✂ } y = f(x) = \frac{1}{3}x^3 - x^2 + x + 10$$

$$\text{✂ } y = f(x) = (2-x)^4$$



Optimization problem and producers

Objective:

$$\pi(q) = TR(q) - TC(q)$$

$$\pi(q) = pq - TC(q)$$

Environment:

TR	TC
A competitive firm $\rightarrow P_{\text{mkt}}$	Production function
A monopolist $\rightarrow \text{inverse demand } P(q)$	Structure of market of factors of productions
An oligopolist	
A monopolistic competition	

Firm's Profit Maximization:

$$\max_q \pi(q)$$

$$\text{FONC : } \frac{d\pi(q)}{dq} = 0$$

$$\frac{d(TR(q) - TC(q))}{dq} = 0$$

$$MR - MC = 0$$

$$MR = MC$$

$\Rightarrow$ find critical value of q
 q^*

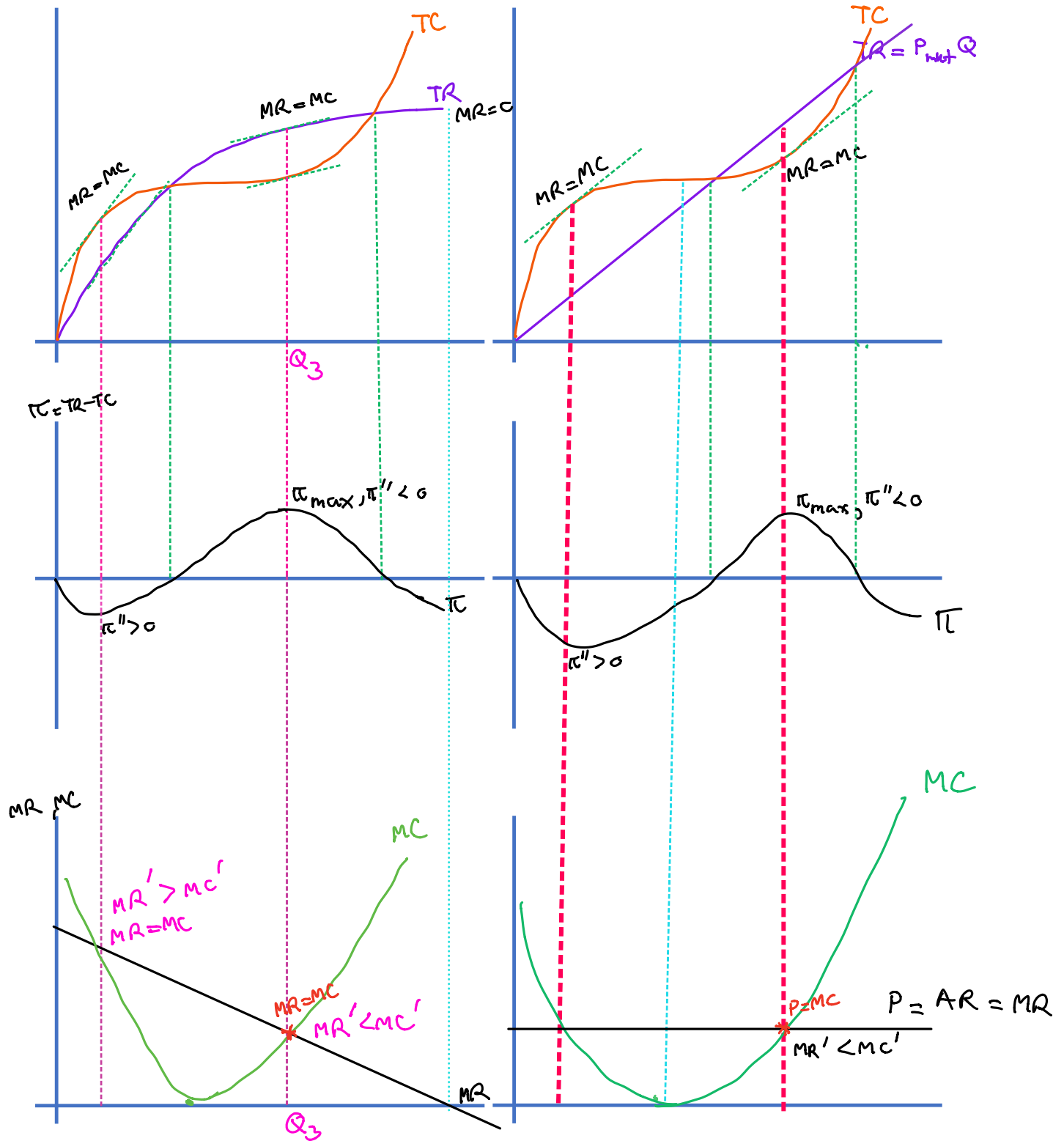
$$\text{SOSC : } \frac{d^2\pi}{dq^2} < 0$$

$$\frac{d(MR - MC)}{dq} = MR' - MC' < 0$$

$MR' < MC' \Rightarrow$ at q^* that gives $\pi_{\max}$
Slope of MR < slope of MC

TR, TC, π , MC, and MR
Monopoly mkt.

Competitive mkt.



➔ Producers in Competitive Market

Recap: Characteristics of competitive market

1. Many Buyers, many sellers
2. Homogeneous product
3. Perfect information among buyers and Sellers
4. Free entry and perfect mobility

* Producers in competitive mkt. are price takers. *

They choose optimal quantity, taking mkt. price as given.

Firm's Profit Maximization:

$$\max_q \pi = p \cdot q - C(q) \quad ; \quad p \text{ is equilibrium mkt. price determined in a market}$$

$$\text{FOC: } \frac{d\pi}{dq} = 0 : p - MC = 0$$

$$p = MC \rightarrow \text{get } q^*$$

$$\text{SGSC: } \frac{d^2\pi}{dq^2} < 0 : \frac{d(p-MC)}{dq} < 0$$

$$-MC' < 0$$

$$MC' > 0 \rightarrow \text{at } q^*, \text{ slope of MC} > 0$$

necessary condition to have maximized profit in a competitive mkt.

$$p = MC$$

Example: Suppose Peter decides to open a company selling shirts in a competitive market. He has total cost $C = 100 + Q^2$, where Q is his level production. If market price for this shirt is 60 baht per shirt, how many shirt should he produce?

$$\max_Q \pi = 60Q - (100 + Q^2)$$

$$\text{FOC: } \frac{d\pi}{dQ} = 60 - 2Q = 0$$

$$Q^* = 30 \quad \#$$

$$\text{SOC: } \frac{d^2\pi}{dQ^2} < 0 \quad ; \quad \frac{d^2\pi}{dQ^2} = -2 < 0$$

$$\therefore \text{ At } Q^*, \pi \text{ is at maximum. } \pi_{\max} = 60(30) - (100 + 30^2) \\ = 1800 - 100 - 900 = 800 \text{ \$}$$

If $Q = 2L$, how many employee should Peter hire so that he can maximize his profit?

$$\max_L \pi = TR(Q(L)) - TC(Q(L))$$

$$\frac{d\pi}{dL} = \frac{dTR}{dQ} \frac{dQ}{dL} - \frac{dTC}{dQ} \frac{dQ}{dL} = 0$$

$$MR \times MP_L - MC \times MP_L = 0$$

$$MRP_L = MFC_L$$

$$\max_L \pi = P \cdot Q(L) - C(Q(L))$$

$$\max_L \pi = 60(2L) - (100 + (2L)^2)$$

$$\max_L \pi = 120L - 100 - 4L^2$$

$$\text{FOC: } \frac{d\pi}{dL} = 120 - 8L = 0 \Rightarrow L^* = 15$$

$$\text{SOC: } \frac{d^2\pi}{dL^2} < 0 \quad ; \quad \frac{d^2\pi}{dL^2} = -8 < 0 \quad \therefore \text{ at } L^* = 15, \\ \pi_{\max} = 120(15) - 100 - 4(15)^2 \\ = 800$$

➤ A monopolist

Recap: Characteristics of monopoly market

1. One producer
2. Inelastic market demand
3. Economy of scale in production
4. Barrier to entry

- * The monopolist has market power to choose price or quantity sold in the mkt.
- * Demand that firm considers is market demand.

Let market demand be $Q = a - bP$.

$$\Rightarrow \text{inverse demand : } P = \frac{a}{b} - \frac{1}{b}Q$$

Total Revenue: TR

$$\begin{aligned} TR &= P(Q)Q, & AR &= \frac{TR}{Q} = P(Q) \\ &= \left(\frac{a}{b} - \frac{1}{b}Q\right)Q, & MR &= \frac{a}{b} - \frac{2}{b}Q \\ TR &= \frac{a}{b}Q - \frac{1}{b}Q^2, & MR &= P\left(1 + \frac{1}{E_p}\right) \end{aligned}$$

Total Cost: TC

$$TC = C(Q)$$

Objective:

$$\pi = TR - TC$$

Find Q that maximizes profit

$$\max_Q \pi = \frac{a}{b}Q - \frac{1}{b}Q^2 - C(Q)$$

FONC:

$$\frac{d\pi}{dQ} = 0 : \frac{d\pi}{dQ} = \overbrace{\frac{a}{b} - \frac{2}{b}Q}^{MR} - C'(Q) = 0$$

$$MR = MC \rightarrow Q^*$$

SOSC:

$$\frac{d^2\pi}{dQ^2} < 0 ; \quad -\frac{2}{b} - C''(Q) < 0$$

$$\underbrace{-\frac{2}{b}}_{MR'} < MC'(Q)$$

Rule of Thumb for Pricing:

$$MC = MR = P \left(1 + \frac{1}{E_p} \right)$$

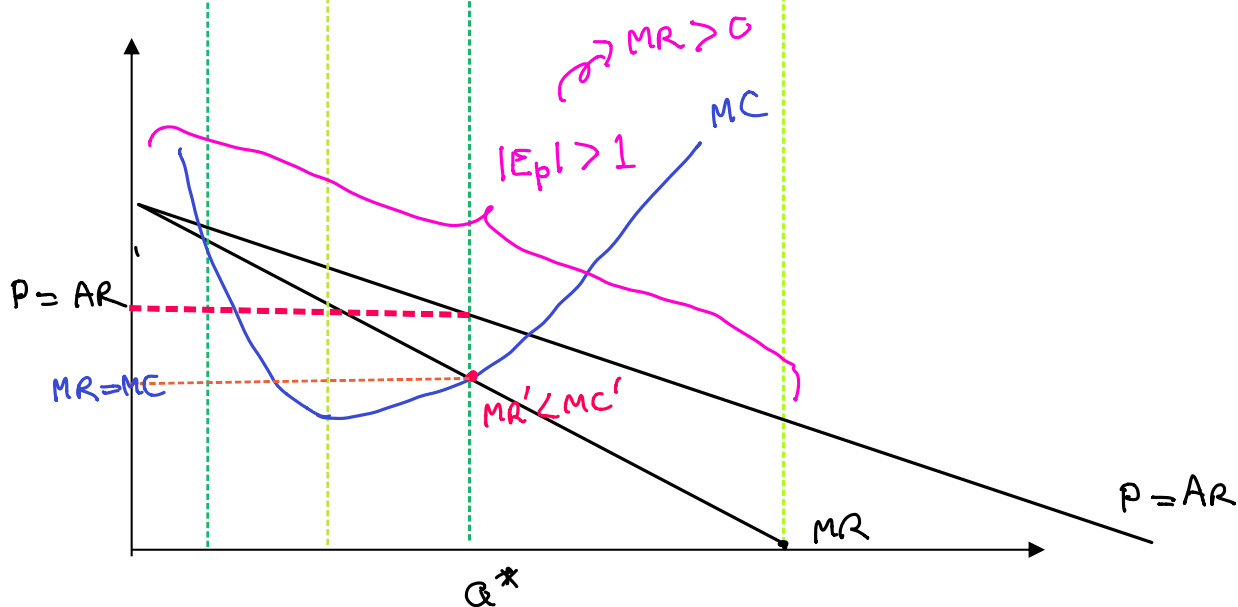
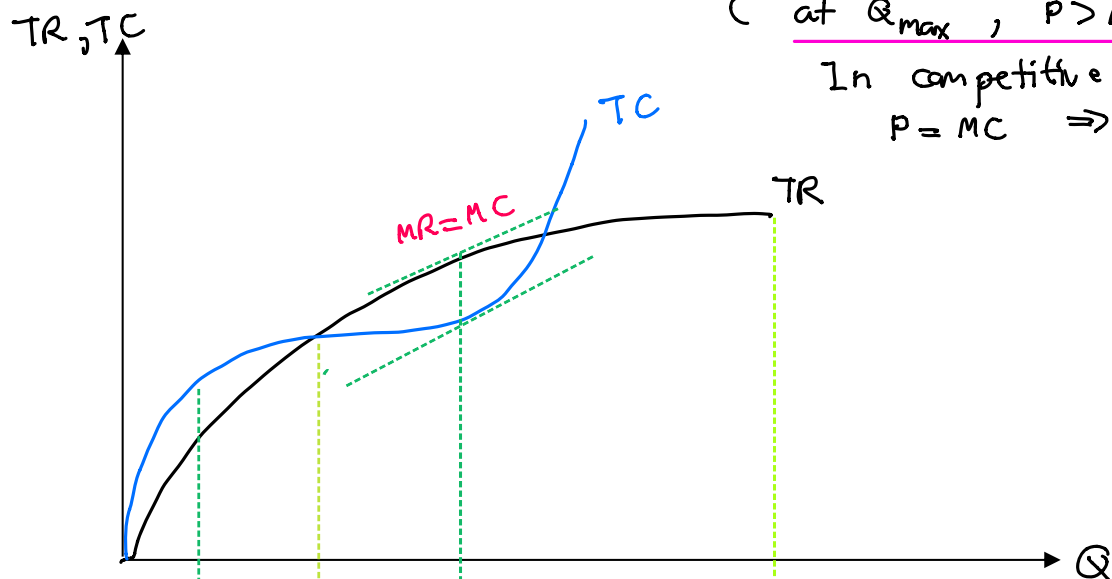
$$\frac{MC - P}{P} = \frac{1}{E_p}$$

$$L = \frac{P - MC}{P} = -\frac{1}{E_p}$$

$L = \text{"Lerner Index"}$

$$0 < L < 1$$

(at Q_{max} , $P > MC$) since $P = AR$
 $AR > MR$
 In competitive mkt.,
 $P = MC \Rightarrow L = 0$
 when $MR > 0$



Measuring Monopoly Power

Lerner's Degree of Monopoly Power

$$L = \frac{(P - MC)}{P} = -\frac{1}{E_p}$$

The Lerner Index is between 0 and 1 since the range of demand that a monopolist chooses to produce has $1 < |E_p| < \infty$

For producers in a competitive market, Lerner Index is 0, since $P=MC$.

The higher the Lerner index is, the higher the monopoly power is.



The effect of tax on profit maximization

Before tax:

$$\pi_{BT} = TR - TC$$

After tax:

$$\pi_{AT} = TR - TC - T, \quad T = \text{Tax}$$

Three types of tax:

Lumsum tax

$$T = \bar{T} \quad \therefore \pi_{AT} = TR - TC - \bar{T}$$

$$\frac{d\pi_{AT}}{dq} = MR - MC = 0$$

$$MR = MC \quad \text{Same condition as}$$

Profit tax:

$$T = t\pi \quad \therefore \pi_{AT} = (1-t)(TR - TC)$$

$$\frac{d\pi_{AT}}{dq} = (1-t)(MR - MC) = 0$$

$$MR - MC = 0 \quad \text{Same condition as}$$

before tax

$$\pi_{AT}^* < \pi_{BT}^*$$

Specific tax:

$$T = tQ \quad \therefore \quad \pi_{AT} = TR - TC - tQ$$

$$\frac{d\pi_{AT}}{dQ} = MR - MC - t = 0$$

$$MR = MC + t$$

different condition than
before tax $\Rightarrow Q_{BT}^* \neq Q_{AT}^*$

Let market demand be

$$P = a - bQ$$

For a monopolist,

$$TR = P(Q)Q = aQ - bQ^2$$

Let total cost be:

$$TC = C(Q) = cQ + f$$

$$\pi_{BT}(Q) = aQ - bQ^2 - cQ - f$$

$$\text{FOC: } \frac{d\pi_{BT}}{dQ} = a - 2bQ - c = 0$$

$$Q_{BT}^* = \frac{a-c}{2b}$$

$$\text{SOC: } \frac{d^2\pi_{BT}}{dQ^2} < 0$$

$$\frac{d^2\pi_{BT}}{dQ^2} = -2b < 0 \quad \therefore \text{ at } Q_{BT}^* = \frac{a-c}{2b}, \pi_{BT} \text{ is maximized}$$

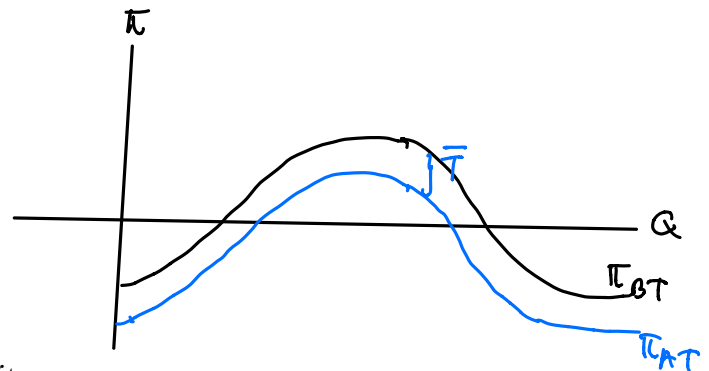
$$P_{BT}^* = a - bQ_{BT}^* = a - b\left(\frac{a-c}{2b}\right) = \frac{a+c}{2}$$

$$\pi_{BT} = \left(\frac{a+c}{2}\right)\left(\frac{a-c}{2b}\right) - c\left(\frac{a-c}{2b}\right) - f$$

Case 1: Lump sum Tax $T = \bar{T}$

Total Profit after Tax

$$\max_Q \pi = \pi_{AT} = TR - TC - \bar{T}$$



Find the level of output that maximizes profit:

$$\text{FONC: } \frac{d\pi_{AT}}{dQ} = \frac{d(TR - TC - \bar{T})}{dQ}$$

$$= MR - MC - 0 = 0$$

$$MR = MC$$

$$\pi_{AT} = aQ - bQ^2 - cQ - f - \bar{T}$$

$$\frac{d\pi_{AT}}{dQ} = a - 2bQ - c = 0$$

$$Q_{AT}^* = \frac{a - c}{2b}$$

$$P_{AT}^* = a - b\left(\frac{a - c}{2b}\right) = \frac{a + c}{2}$$

$$\pi_{AT}(Q) = aQ_{AT}^* - bQ_{AT}^{*2} - cQ_{AT}^* - f - \bar{T}$$

$$\text{SOSC: } \frac{d^2\pi_{AT}}{dQ^2} = \frac{d(MR - MC)}{dQ} = MR' - MC' < 0$$

$$MR' < MC'$$

From FONC, at the level of output at maximizing profit

$$Q_{AT}^* = Q_{BT}^*$$

The profit-maximizing price is

$$p_{AT}^* = a - bQ_{AT}^* = p_{BT}^*$$

Profit after tax is

$$\pi_{AT}^* = \pi_{BT}^* - \bar{T}$$

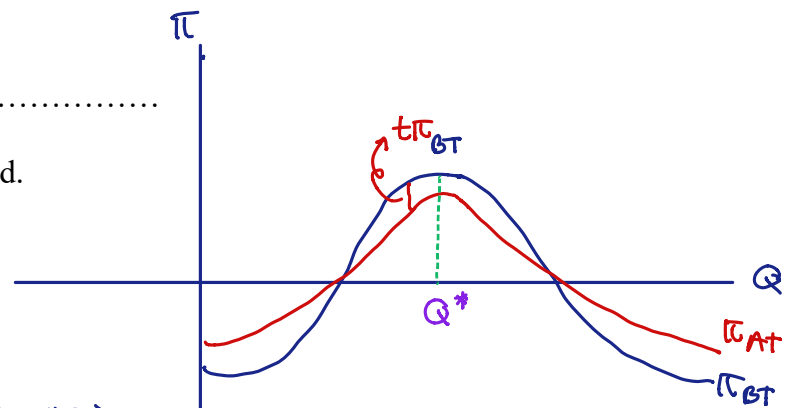
Therefore, if lumpsum tax is collected, the optimal quantity and price will be as same as before tax, but the level of profit will be reduced by the tax amount... $\bar{T}$ Lumpsum tax can be called neutral tax.

Case 2 : Profit Tax $T = t\pi_{BT}$

Suppose tax rate t % of the profit is collected.

Total Profit after Tax:

$$\begin{aligned}\pi_{AT} &= \pi_{BT} - t\pi_{BT} \\ &= (1-t)\pi_{BT} = (1-t)(TR - TC)\end{aligned}$$



Find the level of output that maximizes profit:

$$\begin{aligned}\max_Q \quad \pi_{AT} &= (1-t)(TR - TC) \\ \pi_{AT} &= (1-t)(aQ - bQ^2 - cQ - f)\end{aligned}$$

$$\begin{aligned}\text{FOC:} \quad \frac{d\pi_{AT}}{dQ} &= (1-t)(MR - MC) = 0 \\ MR &= MC\end{aligned}$$

$$\begin{aligned}\frac{d\pi_{AT}}{dQ} &= (1-t)(a - 2bQ - c) = 0 \\ Q_{AT}^* &= \frac{a-c}{2b} = Q_{BT}^* \\ p_{AT}^* &= a - bQ_{AT}^* = p_{BT}^* \\ \pi_{AT} &= (1-t)\pi_{BT}\end{aligned}$$

$$\begin{aligned}\text{SOC:} \quad \frac{d^2\pi_{AT}}{dQ^2} &= (1-t)(MR' - MC') < 0 \\ MR' &< MC', \quad 0 < t < 1\end{aligned}$$

From FONC, at the level of output at maximizing profit

$$Q_{AT}^* = Q_{BT}^*$$

The profit-maximizing price is

$$P_{AT}^* = a - b Q_{AT}^* = P_{BT}^*$$

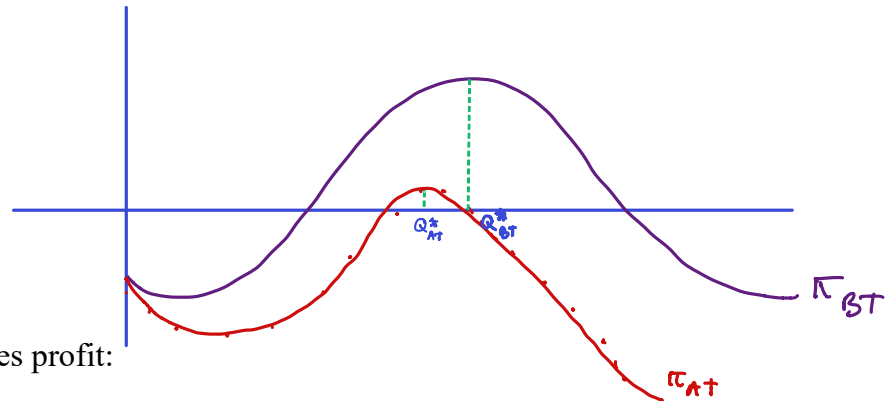
Profit after tax is

$$\pi_{AT} = (1-t)\pi_{BT}$$

Therefore, if lumpsum tax is collected, the optimal quantity and price will be as same as before tax, but the level of profit will be reduced by the tax amount..... $t\pi_{BT}^*$ Profit tax can be called neutral tax.

Case 3: Unit Tax..... tQ Collect t baht per unit (Specific Tax)**Total Profit after Tax:**

$$\pi = \pi_{AT} = TR - TC - tQ$$



Find the level of output that maximizes profit:

$$\max_Q \pi_{AT} = TR - TC - tQ$$

$$\text{FONC: } \frac{d\pi_{AT}}{dQ} = MR - MC - t = 0$$

$$\boxed{\begin{array}{l} MR = MC + t \rightarrow Q_{AT}^* \\ \text{or} \\ MR - t = MC \end{array}}$$

$$\pi_{AT} = aQ - bQ^2 - cQ - f - tQ$$

$$\frac{d\pi_{AT}}{dQ} = a - 2bQ - c - t = 0$$

$$Q_{AT}^* = \frac{a - c - t}{2b} \neq Q_{BT}^* = \frac{a - c}{2b}$$

$$P_{AT}^* = a - b \left(\frac{a - c - t}{2b} \right)$$

$$= \frac{a}{2} + \frac{c}{2} + \frac{t}{2} \neq P_{BT}^* = \frac{a + c}{2}$$

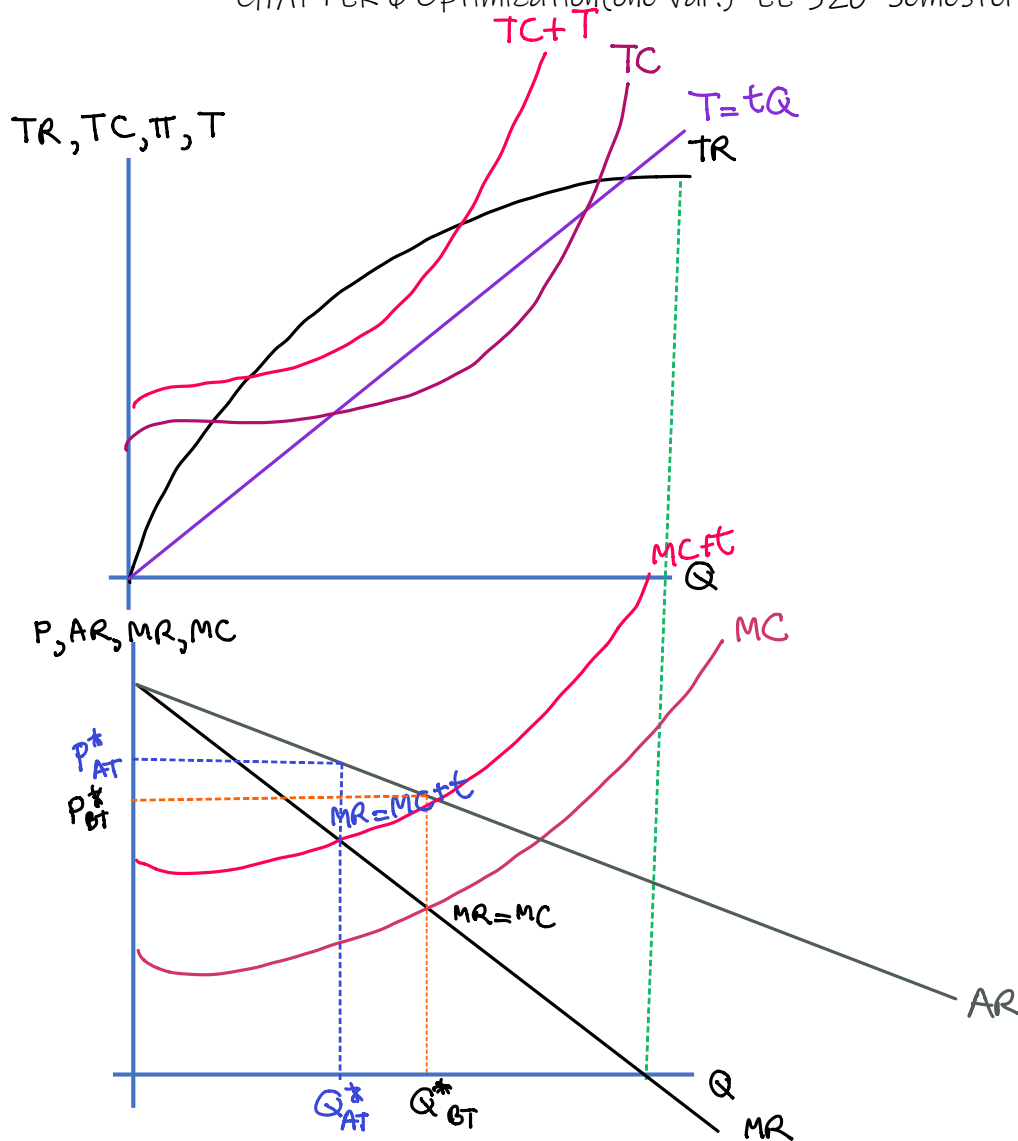
$$\pi_{AT}^* = \left(\frac{a}{2} + \frac{c}{2} + \frac{t}{2} \right) \left(\frac{a - c - t}{2b} \right) - c \left(\frac{a - c - t}{2b} \right) - f - t \left(\frac{a - c - t}{2b} \right) \neq$$

$$\text{SOSC: } \frac{d^2\pi_{AT}}{d^2Q} = MR' - MC' < 0$$

$$MR' < MC'$$

$$\frac{d^2\pi_{AT}}{d^2Q} = -2b < 0, b > 0$$

$\therefore$ at Q_{AT}^* , we get $\pi_{AT, \max}^*$



From FONC, at the level of output at maximizing profit

$$Q^*_{AT} < Q^*_{BT}$$

$\therefore$ we have different FONC after tax

The profit-maximizing price is

$$P^*_{AT} > P^*_{BT}$$

Profit after tax is

$$\pi^*_{AT} = TR(Q^*_{AT}) - TC(Q^*_{AT}) - tQ^*_{AT}$$

If unit tax is collected, producer will reduce production and increase price. The level of profit will be reduced. Unit tax is distortion tax.

H.W.: Comparative Static analysis can be done to see the effect of increase in tax rate by figuring out $\frac{\partial Q^*}{\partial t}$, $\frac{\partial P^*}{\partial t}$ and $\frac{\partial \pi^*}{\partial t}$



Tax Revenue maximization

The next question is which unit tax rate will maximize government's tax revenue.

Profit-maximizing output $Q^* = \frac{a-c-t}{2b}$

With unit tax rate t baht per unit, total tax revenue will be

$$\text{Total Tax Revenue} = t \cdot Q^* = t \left[\frac{a-c-t}{2b} \right] = \frac{(a-c)t - t^2}{2b}$$

$$\max_t T = \frac{(a-c)t - t^2}{2b}$$

FONC $\frac{dT}{dt} = \frac{(a-c) - 2t}{2b} = 0$

$$t^* = \frac{a-c}{2} \quad \#$$

SOSC $\frac{d^2 T}{dt^2} = \frac{-2}{2b} = -\frac{1}{b} < 0$

$\therefore t^*$ is consistent with $T_{\max}$

$$T_{\max} = t^* Q^* = \frac{(a-c)t^* - t^{*2}}{2b}$$

$$= \frac{(a-c)^2}{4b} - \frac{1}{2b} \frac{(a-c)^2}{4}$$

$$= \frac{(a-c)^2}{8b} \quad \#$$