Quiz 1: Date: April 19, 2022 from 11.00-12.30

Question 1 (10 Points) Score.....................

Consider the one-period model of consumption and portfolio choice. Let an individual in this economy has the utility function as follow:

$$U(C) = ln(C)$$

Also, let $\frac{C_1}{C_0}$ is distributed as log-normal with mean equals μ_c and its variance is σ_c.
Please read and answer the following questions carefully and completely.

Score.....................

Question 1.1 (10 marks) Calculate the risk free rate R_f in terms of the individual's consumption, C_0 and C_1. Then,explain the relationship between the level of consumption and the risk free rate in this economy.

maximize $\ln(C_0) + E[\delta \ln(C_1)]$

$C_1 = y_1 + (w_0 + y_0 - C_0)\sum w_i R_i$

$$\underset{C_0, w_i}{L} = U(C_0) + \delta E[U(C_1)] + \lambda[1 - \sum w_i]$$

$$= U(C_0) + \delta E[U(y_1 + (w_0 + y_0 - C_0)\sum w_i R_i)] + \lambda'[1 - \sum w_i]$$

FOC

$$\frac{\partial L}{\partial C_0} = U'(C_0) + \delta E\left[U'(C_1)\left(\sum w_i R_i\right)\right](-1) = 0 \quad - (1)$$

$$\frac{\partial L}{\partial w_i} = \delta E\left[U'(C_1)(w_0 + y_0 - C_0)R_i\right] - \lambda' = 0 \quad - (2)$$

Rearrange (2), $\delta E\left[U'(C_1)R_i\right] = \dfrac{\lambda'}{w_0 + y_0 - C_0} = \lambda$

$$\delta E\left[U'(C_1)R_1\right] = \delta E\left[U'(C_1)R_2\right]$$

$U'(C_0) = \delta E[U'(C_1)\sum w_i R_i] = \sum w_i \delta E[U'(C_1)R_i]$

$= \sum w_i \lambda = \lambda$ since $\sum w_i = 1$

Then, $\lambda = U'(C_0)$

$\delta E[U'(C_1)R_i] = U'(C_0)$ for risky asset

For risk-free asset,

$$R_f \delta E[U'(C_1)] = U'(C_0)$$

$$\frac{1}{R_f} = \delta E\left[\frac{U'(C_1)}{U'(C_0)}\right]$$

With $U(C) = \ln(C)$

$$\frac{1}{R_f} = \delta E\left[\frac{C_0}{C_1}\right]$$

$$R_f = \frac{1}{\delta}\left[\frac{C_1}{C_0}\right] \quad \#$$

Score......................

Question 1.2 (10 marks) Calculate the elasticity of intertemporal substitution in this setting. If in the next year, the interest rate is falling, Will the individual's consumption level increase or decrease? Why? To support your answer, use the concepts of income effect and substitution effect.

$$\frac{\partial R_f}{\partial \frac{C_1}{C_0}} = \frac{1}{\delta}\left[\frac{C_1}{C_0}\right]^{1-1}$$

$$= (1)\frac{R_f}{\frac{C_1}{C_0}}$$

$$\epsilon = \frac{R_f}{C_1/C_0}\frac{\partial C_1/C_0}{\partial R_f} = \frac{\partial \ln(C_1/C_0)}{\partial \ln(R_f)} = 1$$

To support my answer, with no γ and $\epsilon = 1$, the income and substitution effects exactly offset each other. It means that if in the next year, the interest rate is falling, the individual's consumption level remains constant.

Score.....................

Question 1.3 (10 marks) Solve for the pricing kernel P_i of any risky asset i in this economy. Then explain the meaning of this pricing kernel.

From $\delta E[u'(C_1)R_i] = u'(C_0)$

since, $R_i = x_i / P_i$

Then, $P_i u'(C_0) = \delta E[u'(C_1) x_i]$

Rewrite, $P_i = E\left[\frac{\delta u'(C_1)}{u'(C_0)} x_i\right]$

$= E[m_{01} x_i]$

where $m_{01} = \frac{\delta u'(C_1)}{u'(C_0)}$

Utility depends on real consumption, C_1. P_i^N and X_i^N are the initial price and end-of-period payoff measured in currency units (nominal terms), to be deflated by a price index to convert to real quantities.

Real Pricing Kernel

$$\frac{P_i^N}{CPI_0} = E\left[\frac{\delta u'(C_1)}{u'(C_0)} \frac{X_i^N}{CPI_1}\right]$$

$$P_i^N = E\left[\frac{1}{I_{01}} \frac{\delta u'(C_1)}{u'(C_0)} X_i^N\right]$$

$$= E[M_{01} X_i^N]$$

where $M_{01} = \frac{\delta}{I_{01}} \frac{u'(C_1)}{u'(C_0)}$

the stochastic discount factor for nominal returns, equal to the real pricing kernel, m_{01}, discounted at the random rate of inflation between dates 0 and 1.

Score....................

Question 1.4 (10 marks) Calculate Hansen-Jaganathan Bound and explain the meaning.

From $P_i = E[m_{01} x_i]$

Divide each side by P_i

$$1 = E[m_{01} R_i]$$

$$= E[m_{01}]E[R_i] + Cov[m_{01}, R_i]$$

$$= E[m_{01}]\left(E[R_i] + \frac{Cov[m_{01}, R_i]}{E[m_0]}\right)$$

Refer from question 1, $E\left[\frac{\delta u'(c_1)}{u'(c_0)}\right] = E[m_{01}] = \frac{1}{R_f}$

Then, $R_f = E[R_i] + \frac{Cov[m_{01}, R_i]}{E[m_0]}$

$$E[R_i] = R_f - \frac{Cov[m_{01}, R_i]}{E[m_0]}$$

$$= R_f - \frac{Cov[u'(c_1), R_i]}{E[u'(c_1)]} \quad ; u'(c_0) \text{ is cancelled out}$$

$m_{01} = \frac{\delta u'(c_1)}{u'(c_0)}$ and $Cov[m_{01}, R_i] = \rho_{m_{01}, R_i} \sigma_{m_{01}} \sigma_{R_i}$

Then, $E[R_i] = R_f - \rho_{m_{01}, R_i} \frac{\sigma_{m_{01}} \sigma_{R_i}}{E[m_{01}]}$

$$\frac{E[R_i] - R_f}{\sigma_{R_i}} = -\rho_{m_{01}, R_i} \frac{\sigma_{m_{01}}}{E[m_{01}]}$$

Since $-1 \leq \rho_{m_{01}, R_i} \leq 1$

$$\left|\frac{E[R_i] - R_f}{\sigma_{R_i}}\right| = \frac{\sigma_{m_{01}}}{E[m_{01}]} = \sigma_{m_{01}} R_f$$

If there exists a portfolio of assets whose return is perfectly correlated with m_{01}, then the equation holds with equality. The CAPM implies such a situation, so that the slope of the capital market line, $S_e = \frac{E[R_m] - R_f}{\sigma_{R_m}}$, equals $\sigma_{m_{01}} R_f$. According to HJ-Bounds, it means that the maximum premium an investors could get from the market should not exceed $\sigma_{m_{01}} R_f$.