# Chapter 7 Functions of Several Variables

## 7.1 Functions of Two or more Variables

*Notation and Terminology*

There are familiar formulas in which a given variable depends on two or more other variables. For example,

- Area $A$ of a triangle depends on the base length $b$ and height $h$ by the formula $A = \frac{1}{2}bh$.
- The volume $V$ of a rectangular box depends on the length $l$, the width $w$, and the height $h$ by the formula $V = lwh$.
- The arithmetic average $\bar{x}$ of $n$ real numbers, $x_1, x_2, ..., x_n$, depends on those numbers by the formula $\bar{x} = \frac{1}{n}(x_1 + x_2 + ... + x_n)$.

We say that

- $A$ is a function of the two variables $b$ and $h$.
- $V$ is a function of the three variables $l$, $w$, and $h$
- $\bar{x}$ is a function of the $n$ variables $x_1, x_2, ..., x_n$.

The terminology and notation for functions of two or more variables is similar to that for functions of one variable. For example,

$$z = f(x, y)$$

means that $z$ is a function of $x$ and $y$ in the sense that unique value of the dependent variable $z$ is determined by specifying values for the independent variables $x$ and $y$.

**7.1.1 DEFINITION** *A* ***function f of two variables****, x and y, is a rule that assigns a unique real number $f(x, y)$ to each point $(x, y)$ in some set D, the domain of f, in the xy-plane.*

**7.1.2 DEFINITION** *A* ***function f of three variables****, x, y, and z, is a rule that assigns a unique real number $f(x, y, z)$ to each point $(x, y, z)$ in some set D, the domain of f, in three dimensional space.*

**Example 1**: Let $f(x, y) = \sqrt{y+1} + \ln\left(x^2 - y\right)$. Find $f(e, 0)$ and sketch the natural domain of *f*.

**Example 2:** Let $f(x, y, z) = \sqrt{1 - x^2 - y^2 - z^2}$ . Find $f\left(0, \frac{1}{2}, -\frac{1}{2}\right)$.

# 7.2 Partial Derivatives

## 7.2.1 *Partial Derivatives of Functions of Two Variables*

**Definition**: If $z = f(x, y)$ and $(x_0, y_0)$ is a point in the domain of $f$, then the ***partial derivative of f with respect to x*** at $(x_0, y_0)$ [ also called the ***partial derivative of z with respect to x*** at $(x_0, y_0)$ ] is the derivative at $x_0$ of the function that results when $y = y_0$ is held fixed and $x$ is allowed to vary. This partial derivative is denoted by $f_x(x_0, y_0)$ and is given by

$$f_x(x_0, y_0) = \frac{d}{dx}\left[ f(x, y_0) \right]\Big|_{x=x_0}$$

Similarly, the ***partial derivative of f with respect to y*** at $(x_0, y_0)$ [ also called the ***partial derivative of z with respect to y*** at $(x_0, y_0)$ ] is the derivative at $y_0$ of the function that results when $x = x_0$ is held fixed and $y$ is allowed to vary. This partial derivative is denoted by $f_y(x_0, y_0)$ and is given by

$$f_y(x_0, y_0) = \frac{d}{dy}\left[ f(x_0, y) \right]\Big|_{y=y_0}$$

**Notations for Partial Derivatives**

If $z = f(x, y)$, we write

$$f_x(x, y) = f_x = \frac{\partial f}{\partial x} = \frac{\partial}{\partial x} f(x, y) = \frac{\partial z}{\partial x} = D_x f$$

$$f_y(x, y) = f_y = \frac{\partial f}{\partial y} = \frac{\partial}{\partial y} f(x, y) = \frac{\partial z}{\partial y} = D_y f$$

**Example 1**: Find $f_x(1,3)$ and $f_y(1,3)$ for the function $f(x, y) = 2x^3y^2 + 2y + 4x$.

**Example 2**: If $f(x,y) = x^3 + x^2y^3 - 2y^2$, find $f_x(2,1)$ and $f_y(2,1)$.

**Example 3**: Find $\partial z/\partial x$ and $\partial z/\partial y$ if $z = x^4 \sin(xy^3)$.

**Example 4**: If $f(x,y) = x^3 + x^2y^3 - 2y^2$, find $\partial f/\partial x$ and $\partial f/\partial y$

*7.2.2 Partial Derivatives Viewed as Rates of Change and Slopes*

Recall that if $y = f(x)$, then the value of $f'(x_0)$ can be interpreted either as the rate of change of $y$ with respect to $x$ at $x_0$ or as the slope of the tangent line to the graph of $f$ at $x_0$. Partial derivatives have analogous interpretations.

**Example 5**: The wind chill temperature index is given by the formula

$$W = 35.74 + 0.6215T + (0.4275T - 35.75)v^{0.16}$$

Compute the partial derivative of $W$ with respect to $v$ at the point $(T, v) = (25, 10)$ and interpret this partial derivative as a rate of change.

*7.2.3 Implicit Partial Differentiation*

**Example 8**: Find the slope of the sphere $x^2 + y^2 + z^2 = 1$ in the *y*-direction at the points $\left(\frac{2}{3}, \frac{1}{3}, \frac{2}{3}\right)$ and $\left(\frac{2}{3}, \frac{1}{3}, -\frac{2}{3}\right)$.

**Example 9**: Suppose that $D=\sqrt{x^2+y^2}$ is the length of the diagonal of a rectangle whose sides have lengths $x$ and $y$ that are allowed to vary. Find a formula for the rate of change of *D* with respect to $x$ if $x$ varies with $y$ held constant, and use this formula to find the rate of change of *D* with respect to *x* at the point where *x* = 3 and *y* = 4.

**Example 10**: Find $\partial z/\partial x$ and $\partial z/\partial y$ if $z$ is defined implicitly as a function of $x$ and $y$ by the equation

$$x^3+y^3+z^3+6xyz=1$$

*7.2.4 Partial Derivatives of Functions with more than Two Variables*

For a function $f(x, y, z)$ of three variables, there are three partial derivatives:

$$f_x(x, y, z), \quad f_y(x, y, z), \quad f_z(x, y, z)$$

If a dependent variable

$$w = f(x, y, z)$$

is used, then the three partial derivatives of $f$ can be denoted by

$$\frac{\partial w}{\partial x}, \quad \frac{\partial w}{\partial y}, \quad \frac{\partial w}{\partial z}$$

**Example 11**: If $f(x, y, z) = x^3y^2z^4 + 2xy + z$. Find $f_x(x, y, z)$, $f_y(x, y, z)$, $f_z(x, y, z)$, and $f_z(-1, 1, 2)$.

**Example 12**: Find $f_x$, $f_y$, and $f_z$ if $f(x, y, z) = e^{xy} \ln z$

*7.2.5 Higher-Order Partial Derivatives*

Suppose that $f$ is a function of two variable $x$ and $y$. Since the partial derivatives $\partial f / \partial x$ and $\partial f / \partial y$ are also functions of $x$ and $y$, these functions may themselves have partial derivatives. This gives rise to four possible ***second-order*** partial derivatives of $f$, which are defined by

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = f_{xx} \qquad \frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = f_{yy}$$

$$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = f_{xy} \qquad \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = f_{yx}$$

The last two cases are called the ***mixed second-order partial derivatives*** or the ***mixed second partials.***

**Example 13**: Let $f(x, y) = y^2 e^x + y$. Find $\dfrac{\partial^3 f}{\partial y^2 \partial x}$.

**Example 14**: Find the second partial derivatives of

$$f(x, y) = x^3 + x^2 y^3 - 2y^2$$

Notice that $f_{xy} = f_{yx}$ in Example 14. This is not just a coincidence. It turns out that the mixed partial derivatives $f_{xy}$ and $f_{yx}$ are equal for most functions that one meets in practices.

## 7.3 The Chain Rule

### *7.3.1 Chain Rules for Derivatives*

**Theorem** (Chain Rules for Derivatives): If $x = x(t)$ and $y = y(t)$ are differentiable at $t$, and if $z = f(x, y)$ is differentiable at the point $(x, y) = (x(t), y(t))$, then $z = f(x(t), y(t))$ is differentiable at $t$ and

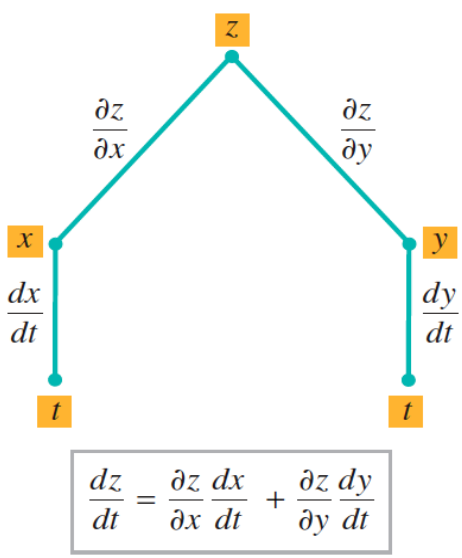
$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}$$

where the ordinary derivatives are evaluated at $t$ and the partial derivatives are evaluated at $(x, y)$.

If each of the functions $x = x(t)$, $y = y(t)$, and $z = z(t)$ is differentiable at $t$, and if $w = f(x, y, z)$ is differentiable at the point $(x, y, z) = (x(t), y(t), z(t))$, then the function $w = f(x(t), y(t), z(t))$ is differentiable at $t$ and

$$\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt} + \frac{\partial w}{\partial z}\frac{dz}{dt}$$

where the ordinary derivatives are evaluated at $t$ and the partial derivatives are evaluated at $(x, y, z)$.



**Example 1**: Suppose that

$$z = x^2 y, \quad x = t^2, \quad y = t^3$$

Use the chain rule to find $dz/dt$, and check the result by expressing $z$ as a function of $t$ and differentiating directly.

**Example 2**: If

$$z = x^2 y + 3xy^4 \quad \text{where} \quad x = \sin 2t, \quad y = \cos t$$

Find $dz/dt$ when $t = 0$.

**Example 3**: Suppose that

$$w = \sqrt{x^2 + y^2 + z^2}, \quad x = \cos\theta, \quad y = \sin\theta, \quad z = \tan\theta$$

Use the chain rule to find $dw/d\theta$ when $\theta = \pi/4$

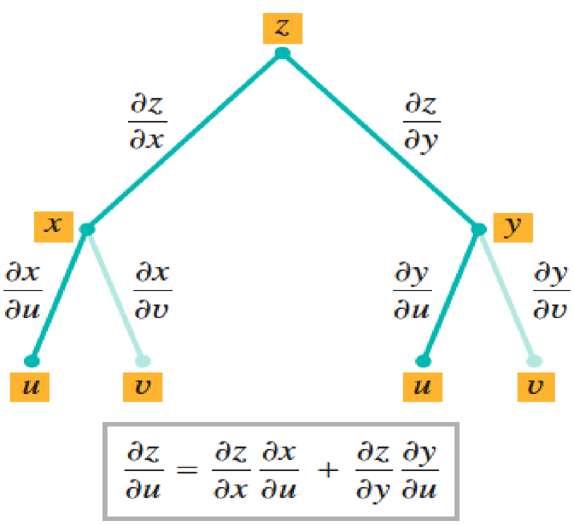
*7.3.2 Chain Rules for Partial Derivatives*

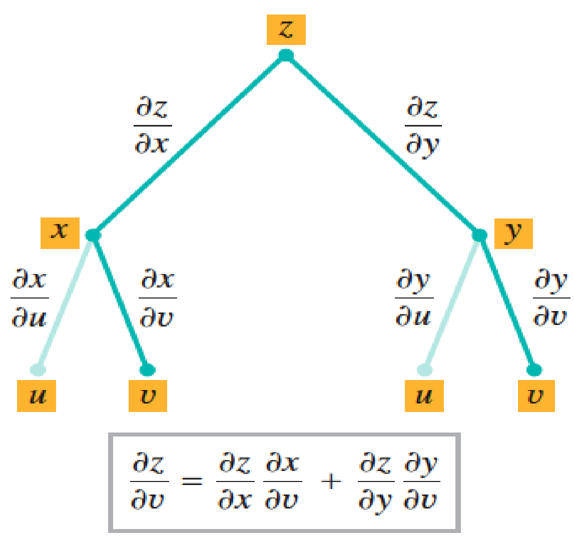
**Theorem** (Chain Rules for Partial Derivatives): If $x = x(u,v)$ and $y = y(u,v)$ have first-order partial derivatives at the point $(u,v)$, and if $z = f(x,y)$ is differentiable at the point $(x,y) = (x(u,v), y(u,v))$, then $z = f(x(u,v), y(u,v))$ has first order partial derivatives at the point $(u,v)$ given by

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial u} \quad \text{and} \quad \frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial v}$$

If each function $x = x(u,v)$, $y = y(u,v)$, and $z = z(u,v)$ has first-order partial derivatives at the point $(u,v)$, and if $w = f(x,y,z)$ is differentiable at the point $(x,y,z) = (x(u,v), y(u,v), z(u,v))$, then $w = f(x(u,v), y(u,v), z(u,v))$ has first order partial derivatives at the point $(u,v)$ given by

$$\frac{\partial w}{\partial u} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial u} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial u} \quad \text{and} \quad \frac{\partial w}{\partial v} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial v} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial v}$$

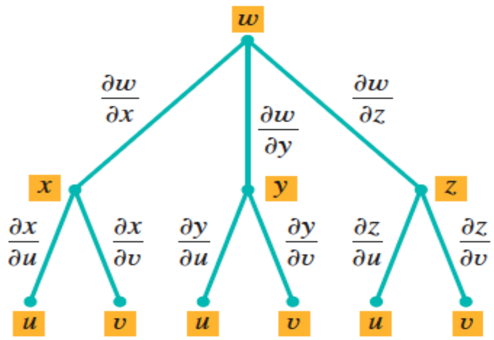


**Example 5**: Given that

$$z = e^{xy}, \quad x = 2u + v, \quad y = u / v$$

Find $\partial z / \partial u$ and $\partial z / \partial v$ using the chain rule.




$$\frac{\partial w}{\partial u} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial u} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial u}$$

$$\frac{\partial w}{\partial v} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial v} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial v}$$

**Example 6**: Suppose that

$$w = e^{xyz}, \quad x = 3u + v, \quad y = 3u - v, \quad z = u^2 v$$

Find $\partial w / \partial u$ and $\partial w / \partial v$ using the chain rule.

**Example 7**: Suppose that $w = x^2 + y^2 - z^2$ and

$$x = \rho \sin\phi \cos\theta, \quad y = \rho \sin\phi \sin\theta, \quad z = \rho \cos\phi$$

Use appropriate forms of the chain rule to find $\partial w / \partial \rho$ and $\partial w / \partial \theta$

**Example 8**: Suppose that

$$w = xy + yz, \quad y = \sin x, \quad z = e^x$$

Use an appropriate form of the chain rule to find $dw / dx$

*7.3.3 Implicit Differentiation*

Consider the special case where $z = f(x, y)$ is a function of $x$ and $y$, and $y$ is a differentiable function of $x$. Equation (5) then becomes

$$\frac{dz}{dx} = \frac{\partial f}{\partial x}\frac{dx}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dx} = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\frac{dy}{dx} \qquad (*)$$

This result can be used to find derivatives of functions that are defined implicitly. For example, suppose that the equation

$$f(x, y) = c \qquad (**)$$

Defines $y$ implicitly as a differentiable function of $x$ and we are interested in finding $dy/dx$. Differentiating both sides of (**) with respect to $x$ and applying (*) yields

$$\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\frac{dy}{dx} = 0$$

Thus, if $\partial f / \partial y \neq 0$, we obtain

$$\frac{dy}{dx} = -\frac{\partial f / \partial x}{\partial f / \partial y}$$

In summary, we have the following result.

**Theorem**: If the equation $f(x, y) = c$ defines $y$ implicitly as a differentiable function of $x$, and if $\frac{\partial f}{\partial y} \neq 0$, then

$$\frac{dy}{dx} = -\frac{\partial f / \partial x}{\partial f / \partial y}$$

**Example 10**: Given that $x^3 + y^2 x - 3 = 0$, find $dy/dx$ using the Theorem and check the result using implicit differentiation.

**Example 11**: Find $dy/dx$ if $x^3 + y^3 = 6xy$ using the Theorem.

The chain rule also applies to implicit partial differentiation. Consider the case where $w = f(x, y, z)$ is a function of $x, y,$ and $z$ and $z$ is a differentiable function of $x$ and $y$. It follows from Theorem that

$$\frac{\partial w}{\partial x} = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial x} \qquad (*)$$

If the equation

$$f(x, y, z) = c \qquad (**)$$

defines $z$ implicitly as a differentiable function of $x$ and $y$, then taking te partial derivative of each side of $(**)$ with respect to $x$ and applying $(*)$ gives

$$\frac{\partial f}{\partial x} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial x} = 0$$

If $\partial f / \partial z \neq 0$, then

$$\frac{\partial z}{\partial x} = -\frac{\partial f / \partial x}{\partial f / \partial z}$$

A similar result holds for $\partial z / \partial y$.

**Theorem**: If the equation $f(x, y, z) = c$ defines $z$ implicitly as a differentiable function of $x$ and $y$, and if $\frac{\partial f}{\partial z} \neq 0$, then

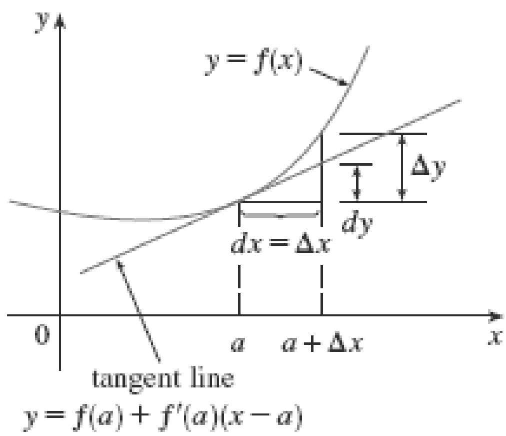
$$\frac{\partial z}{\partial x} = -\frac{\partial f / \partial x}{\partial f / \partial z} \quad \text{and} \quad \frac{\partial z}{\partial y} = -\frac{\partial f / \partial y}{\partial f / \partial z}$$

**Example 12**: Consider the sphere $x^2 + y^2 + z^2 = 1$. Find $\partial z / \partial x$ and $\partial z / \partial y$ at the point $\left(\frac{2}{3}, \frac{1}{3}, \frac{2}{3}\right)$

## 7.4 Total Differential and Its Applications

For a differentiable function of one variable, $y = f(x)$, we define the differential *dx* to be an independent variable; that is, *dx* can be given the value of any real number. The differential of *y* is then defined as

$$dy = f'(x)\,dx$$




The figure above shows the relationship between the increment $\Delta y$ and the differential *dy*: $\Delta y$ represents the change in height of the curve $y = f(x)$ and *dy* represents the change in height of the tangent line when *x* changes by an amount $dx = \Delta x$.

For a differentiable function of two variables, $z = f(x, y)$, we define the differentials *dx* and *dy* to be independent variables; that is, they can be given any values. Then the **differential** *dz*, also called the **total differential**, is defined by

$$\boxed{dz = f_x(x, y)\,dx + f_y(x, y)\,dy = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy} \qquad (*)$$

Sometimes the notation *df* is used in place of *dz*.

If we take $dx = \Delta x = x - a$ and $dy = \Delta y = y - b$ in Equation (*), then the differential of *z* is

$$dz = f_x(a,b)(x-a) + f_y(a,b)(y-b)$$

So, in the notation of differentials, the **linear approximation**

$$f(x,y) \approx f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b)$$

can be written as

$$f(x,y) \approx f(a,b) + dz$$

Note: The **linearization** of $f$ at $(a,b)$ is

$$L(x,y) = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b)$$

**Example 1**:

(a) If $z = f(x,y) = x^2 + 3xy - y^2$, find the differential $dz$.

(b) If $x$ changes from 2 to 2.05 and $y$ changes from 3 to 2.96, compare the values of $\Delta z$ and $dz$.

**Example 2**: Show that $f(x, y) = xe^{xy}$ is differentiable at $(1,0)$ and find its linearization there. Then use it to approximate $f(1.1, -0.1)$.