Exercise Solution: Solving Systems of Linear Equations

1. Find all solution(s) of the following systems of linear equations by using **Gaussian elimination method**.

(a)
$$\begin{array}{rcrcrcrcr} x_1 &+& x_2 &+& x_3 &+& x_4 &=& 6 \\ x_1 &+& 2x_2 &+& 3x_3 &+& 4x_4 &=& 16 \\ 2x_1 &+& 3x_2 &+& 5x_3 &+& 6x_4 &=& 25 \\ x_1 &+& x_2 &+& 2x_3 &+& 3x_4 &=& 11 \end{array}$$

(b)
$$\begin{array}{rcrcrcr} x &+& 2y &+& 3z &=& 1 \\ 3x &+& 2y &+& z &=& 1 \\ 7x &+& 2y &-& 3z &=& 1 \end{array}$$

Answer:

(a) $$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 1 \\ 2 \end{bmatrix}$$

The augmented form is given by $\left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 6 \\ 1 & 2 & 3 & 4 & 16 \\ 2 & 3 & 5 & 6 & 25 \\ 1 & 1 & 2 & 3 & 11 \end{array}\right]$.

Step 1: R_1 is the "*pivot row*" and $a_{11} = 1$ is the "*pivot element*".
Let $m_{21} = \frac{a_{21}}{a_{11}} = \frac{1}{1} = 1$, $m_{31} = \frac{a_{31}}{a_{11}} = \frac{2}{1} = 2$, and $m_{41} = \frac{a_{41}}{a_{11}} = \frac{1}{1} = 1$.

$$\begin{array}{l} R_1 : \\ R_2 \mapsto R_2 - m_{21}R_1 : \\ R_3 \mapsto R_3 - m_{31}R_1 : \\ R_4 \mapsto R_4 - m_{41}R_1 : \end{array} \left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 6 \\ 0 & 1 & 2 & 3 & 10 \\ 0 & 1 & 3 & 4 & 13 \\ 0 & 0 & 1 & 2 & 5 \end{array}\right]$$

Step 2: R_2 is the "*pivot row*" and $a_{22} = 1$ is the "*pivot element*".
Let $m_{32} = \frac{a_{32}}{a_{22}} = \frac{1}{1} = 1$ and $m_{42} = \frac{a_{42}}{a_{22}} = \frac{0}{1} = 0$.

$$\begin{array}{l} R_1 : \\ R_2 : \\ R_3 \mapsto R_3 - m_{32}R_2 : \\ R_4 \mapsto R_4 - m_{42}R_2 : \end{array} \left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 6 \\ 0 & 1 & 2 & 3 & 10 \\ 0 & 0 & 1 & 1 & 3 \\ 0 & 0 & 1 & 2 & 5 \end{array}\right]$$

Step 3: R_3 is the "*pivot row*" and $a_{33} = 1$ is the "*pivot element*".
Let $m_{43} = \frac{a_{43}}{a_{33}} = \frac{1}{1} = 1$.

$$\begin{array}{l} R_1 : \\ R_2 : \\ R_3 : \\ R_4 \mapsto R_4 - m_{43}R_3 : \end{array} \left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 6 \\ 0 & 1 & 2 & 3 & 10 \\ 0 & 0 & 1 & 1 & 3 \\ 0 & 0 & 0 & 1 & 2 \end{array}\right]$$

This gives the following equivalent system.

$$\begin{array}{rcrcrcrcr} x_1 &+& x_2 &+& x_3 &+& x_4 &=& 6 \\ && x_2 &+& 2x_3 &+& 3x_4 &=& 10 \\ &&&& x_3 &+& x_4 &=& 3 \\ &&&&&& x_4 &=& 2 \end{array}$$

By applying back substitution, we have

R_4: $x_4 = 2$

R_3: $x_3 + 2 = 3 \implies x_3 = 1$

R_2: $x_2 + 2(1) + 3(2) = 10 \implies x_2 = 2$

R_1: $x_1 + 2 + 1 + 2 = 6 \implies x_1 = 1$

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(b) From $\begin{array}{rcrcrcr} x & + & 2y & + & 3z & = & 1 \\ 3x & + & 2y & + & z & = & 1 \\ 7x & + & 2y & - & 3z & = & 1 \end{array}$, the augmented form is $\left[\begin{array}{rrr|r} 1 & 2 & 3 & 1 \\ 3 & 2 & 1 & 1 \\ 7 & 2 & -3 & 1 \end{array}\right]$.

Step 1: R_1 is the "*pivot row*" and $a_{11} = 1$ is the "*pivot element*".
Let $m_{21} = \frac{a_{21}}{a_{11}} = \frac{3}{1} = 3$ and $m_{31} = \frac{a_{31}}{a_{11}} = \frac{7}{1} = 7$.

$$\begin{array}{l} R_1 : \\ R_2 \mapsto R_2 - m_{21}R_1 : \\ R_3 \mapsto R_3 - m_{31}R_1 : \end{array} \left[\begin{array}{rrr|r} 1 & 2 & 3 & 1 \\ 0 & -4 & -8 & -2 \\ 0 & -12 & -24 & -6 \end{array}\right]$$

Step 2: R_2 is the "*pivot row*" and $a_{22} = -4$ is the "*pivot element*".
Let $m_{32} = \frac{a_{32}}{a_{22}} = \frac{-12}{-4} = 3$.

$$\begin{array}{l} R_1 : \\ R_2 : \\ R_3 \mapsto R_3 - m_{32}R_2 : \end{array} \left[\begin{array}{rrr|r} 1 & 2 & 3 & 1 \\ 0 & -4 & -8 & -2 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

By applying back substitution, we have

R_3: z can be any real number, say, $z = t, t \in \mathbb{R}$.

R_2: $-4y - 8z = -2 \implies y = \frac{1}{2} - 2t,\ t \in \mathbb{R}$

R_1: $x + 2y + 3z = 1 \implies x = 1 - 2(\frac{1}{2} - 2t) - 3t = t.$

Hence there are infinitely many solutions and can be given by $\left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} t \\ \frac{1}{2} - 2t \\ t \end{bmatrix} \;\middle|\; t \in \mathbb{R} \right\}$.

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Optional Problems

1. Consider the system of 3 equations and 3 unknowns x, y, z:

$$\begin{array}{rcrcrcr} x & + & 2y & + & 3z & = & 1 \\ x & + & 3y & + & 4z & = & 3 \\ x & + & 4y & + & kz & = & m, \end{array}$$

where m and k are some constants.

(a) Suppose $k = 5$. Find the value of m that makes the above system of equations have infinitely many solutions.

(b) Suppose $m = 4$. Find the value of m that makes the above system of equations have no solution.

Answer: By performing Gaussian elimination method, we have:

(a) $m = 5$.

(b) $k = 5$.

2. Solve the following system of linear equations by using **Gaussian elimination method**.

$$\begin{array}{rcrcrcrcrcr} x_1 & + & x_2 & + & & & x_4 & + & x_5 & = & 6 \\ x_1 & + & 2x_2 & + & 3x_3 & + & 4x_4 & + & x_5 & = & 17 \\ 2x_1 & + & 3x_2 & + & 5x_3 & + & & & 6x_5 & = & 19 \\ & + & x_2 & + & x_3 & + & 2x_4 & + & 3x_5 & = & 10 \\ x_1 & + & 2x_2 & + & 2x_3 & + & x_4 & & & = & 9 \end{array}$$

Answer:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 1 \\ 2 \\ 1 \end{bmatrix}.$$

The augmented form is given by $\left[\begin{array}{ccccc|c} 1 & 1 & 0 & 1 & 1 & 6 \\ 1 & 2 & 3 & 4 & 1 & 17 \\ 2 & 3 & 5 & 0 & 6 & 19 \\ 0 & 1 & 1 & 2 & 3 & 10 \\ 1 & 2 & 2 & 1 & 0 & 9 \end{array}\right]$.

Step 1: R_1 is the "*pivot row*" and $a_{11} = 1$ is the "*pivot element*".
Let $m_{21} = \frac{a_{21}}{a_{11}} = \frac{1}{1} = 1$, $m_{31} = \frac{a_{31}}{a_{11}} = \frac{2}{1} = 2$, $m_{41} = \frac{a_{41}}{a_{11}} = \frac{0}{1} = 0$, and $m_{51} = \frac{a_{51}}{a_{11}} = \frac{1}{1} = 1$.

$$\begin{array}{l} R_1 : \\ R_2 \mapsto R_2 - m_{21}R_1 : \\ R_3 \mapsto R_3 - m_{31}R_1 : \\ R_4 \mapsto R_4 - m_{41}R_1 : \\ R_5 \mapsto R_5 - m_{51}R_1 : \end{array} \left[\begin{array}{ccccc|c} 1 & 1 & 0 & 1 & 1 & 6 \\ 0 & 1 & 3 & 3 & 0 & 11 \\ 0 & 1 & 5 & -2 & 4 & 7 \\ 0 & 1 & 1 & 2 & 3 & 10 \\ 0 & 1 & 2 & 0 & -1 & 3 \end{array}\right]$$

Step 2: R_2 is the "*pivot row*" and $a_{22} = 1$ is the "*pivot element*".
Let $m_{32} = \frac{a_{32}}{a_{22}} = \frac{1}{1} = 1$, $m_{42} = \frac{a_{42}}{a_{22}} = \frac{1}{1} = 1$, and $m_{52} = \frac{a_{52}}{a_{22}} = \frac{1}{1} = 1$.

$$\begin{array}{l} R_1 : \\ R_2 : \\ R_3 \mapsto R_3 - m_{32}R_2 : \\ R_4 \mapsto R_4 - m_{42}R_2 : \\ R_5 \mapsto R_5 - m_{52}R_2 : \end{array} \left[\begin{array}{ccccc|c} 1 & 1 & 0 & 1 & 1 & 6 \\ 0 & 1 & 3 & 3 & 0 & 11 \\ 0 & 0 & 2 & -5 & 4 & -4 \\ 0 & 0 & -2 & -1 & 3 & -1 \\ 0 & 0 & -1 & -3 & -1 & -8 \end{array}\right]$$

Step 3: R_3 is the "*pivot row*" and $a_{33} = 2$ is the "*pivot element*".
Let $m_{43} = \frac{a_{43}}{a_{33}} = \frac{-2}{2} = -1$, and $m_{53} = \frac{a_{53}}{a_{33}} = \frac{-1}{2} = -0.5$.

$$\begin{array}{l} R_1 : \\ R_2 : \\ R_3 : \\ R_4 \mapsto R_4 - m_{43}R_3 : \\ R_5 \mapsto R_5 - m_{53}R_3 : \end{array} \left[\begin{array}{ccccc|c} 1 & 1 & 0 & 1 & 1 & 6 \\ 0 & 1 & 3 & 3 & 0 & 11 \\ 0 & 0 & 2 & -5 & 4 & -4 \\ 0 & 0 & 0 & -6 & 7 & -5 \\ 0 & 0 & 0 & -11/2 & 1 & -10 \end{array}\right]$$

Step 4: R_4 is the "*pivot row*" and $a_{44} = -6$ is the "*pivot element*".
Let $m_{54} = \frac{a_{54}}{a_{44}} = \frac{-11/2}{-6} = \frac{11}{12}$.

$$\begin{array}{l} R_1 : \\ R_2 : \\ R_3 : \\ R_4 : \\ R_5 \mapsto R_5 - m_{54}R_4 : \end{array} \left[\begin{array}{ccccc|c} 1 & 1 & 0 & 1 & 1 & 6 \\ 0 & 1 & 3 & 3 & 0 & 11 \\ 0 & 0 & 2 & -5 & 4 & -4 \\ 0 & 0 & 0 & -6 & 7 & -5 \\ 0 & 0 & 0 & 0 & -65/12 & -65/12 \end{array}\right]$$

This gives the following equivalent system.

$$\begin{array}{rcrcrcrcrcr} x_1 & + & x_2 & + & & & x_4 & + & x_5 & = & 6 \\ & & x_2 & + & 3x_3 & + & 3x_4 & + & & = & 11 \\ & & & & 2x_3 & - & 5x_4 & + & 4x_5 & = & -4 \\ & & & & & - & 6x_4 & + & 7x_5 & = & -5 \\ & & & & & & & - & \frac{65}{12}x_5 & = & -\frac{65}{12} \end{array}$$

By applying back substitution, we have

R_5: $-\frac{65}{12}x_5 = -\frac{65}{12} \implies x_5 = 1$

R_4: $-6x_4 + 7(1) = -5 \implies x_4 = 2$

R_3: $2x_3 - 5(2) + 4(1) = -4 \implies x_3 = 1$

R_2: $x_2 + 3(1) + 3(2) = 11 \implies x_2 = 2$

R_1: $x_1 + 2 + 2 + 1 = 6 \implies x_1 = 1$

3. The Russian-born U.S. economist and Novel laureate Wassily Leontief (1906-1999) was interested in the following question:

   What output industries in an economy produce to satisfy the total demand of all products?

   Here, we will consider a very simple example of input-output analysis, an economy with only two industries, A and B. Assume that the consumer demand for their products is, respectively, 310 and 100, in millions of dollars per year.

   Let a and b be the output (in millions of dollars per year) of industries A and B, respectively. What amount of outputs a and b should the two industries generate to satisfy the demand? Ones may be tempted to say 1000 and 780, respectively. However, things are not quite as simple as that. We have to take into account the *interindustry demand* as well. E.g. suppose A produces electricity and B produces mechanical devices. Suppose also that

   - industry B needs \$0.3 worth of electrical power from A for each \$1 of output B produces;
   - industry A needs \$0.5 worth of mechanical devices from B for each \$1 of output A produces.

Construct the corresponding system of linear equations to find the outputs a and b needed to satisfy both consumer and interindustry demand.

Answer: The corresponding equations are $a = 310 + 0.3b$ and $b = 100 + 0.5a$, i.e.

$$\begin{aligned} a - 0.3b &= 310 \\ -0.5a + b &= 100. \end{aligned}$$

Solving this system gives $a = 400, b = 300$.
Remarks: More realistic problems could consist of an extremely large number of unknowns and a systematic way of solving linear systems, e.g. Gaussian elimination method, is generally required.