Solving Systems of Linear Equations

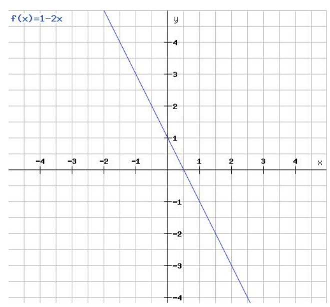
1 Linear equations

We consider a linear equation of the form

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = b,$$

where $a_1, a_2, \ldots, a_n$ are *constants* and $x_1, x_2, \ldots, x_n$ are *variables* or *unknowns.* Notice, this is *linear* in term of the *power* of x_i, $i = 1, \ldots, n$.

- The variables $x_1, x_2, \ldots, x_n$ that solve or satisfy the equation is called **solution** of that equation.
- This equation generally has infinitely many solutions when $n \geq 2$.
- **Example:** for $n = 2$, the equation
$$2x + y = 1$$
has *infinitely many solutions for* (x, y).
  - Two of the solutions are: $x = 1$, $y = -1$ and $x = 0$, $y = 1$
  - Set of all solutions: $\{(x, y) = (a, 1-2a) | a \in \mathbb{R}\}$, which are on the a straight line: $y = 1-2x$.



- In general, if the number of equations is less than the number of unknowns, there will be either no solution or infinitely many solutions.

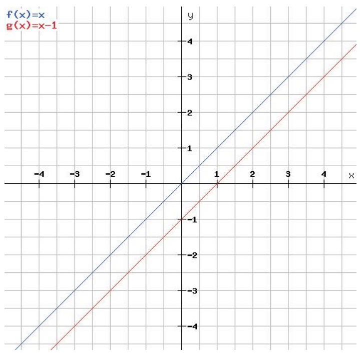
System of linear equations
The general form of system of linear equations with n equations and n unknowns is:

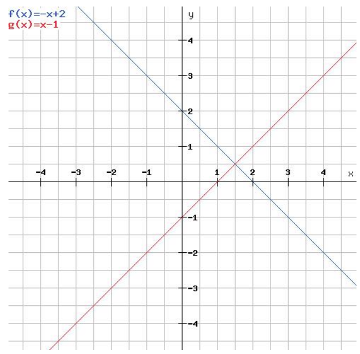
$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ \vdots \quad \vdots &\quad \vdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n &= b_n \end{aligned}$$

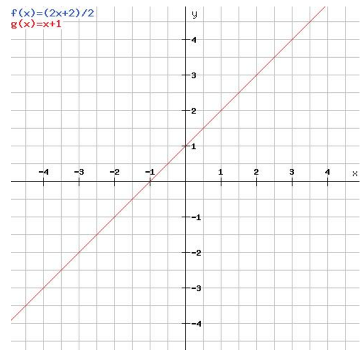
where a_{ij} and b_i are constant real numbers, for $i, j = 1, 2, \ldots, n$; $x_1, \ldots, x_n$ are variables or unknowns.

There are 3 possible cases for the solution of a system of linear equations with n equations and n unknowns.

- There is no solution. E.g. $\begin{cases} -x+y &= 0 \\ -x+y &= -1 \end{cases}$
- There is exactly one unique solution. E.g. $\begin{cases} x+y &= 2 \\ -x+y &= -1 \end{cases}$
- There are many solutions. E.g. $\begin{cases} -2x+2y &= 2 \\ -x+y &= 1 \end{cases}$







- The general form of system of linear equations is:

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ \vdots \quad &\vdots \quad \vdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n &= b_n \end{aligned}$$

  where a_{ij} and b_i are some constant numbers, for $i, j = 1, 2, \ldots, n$.

- This system can be written in the matrix-vector form as:

$$\mathbf{Ax} = \mathbf{b} \quad \text{where} \quad \mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \vdots & \vdots & \vdots & \\ a_{n1} & a_{n2} & \ldots & a_{nn} \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}.$$

- **Augmented matrix** of the above linear system is given by

$$[A|b] \quad \text{or} \quad \left[\begin{array}{cccc|c} a_{11} & a_{12} & \ldots & a_{1n} & b_1 \\ a_{21} & a_{22} & \ldots & a_{2n} & b_2 \\ \vdots & \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \ldots & a_{nn} & b_n \end{array}\right].$$

Example 1.1. Equivalent forms of linear system.

$$\left.\begin{array}{lccccccc} (R_1) & a_{11}x_1 & + & a_{12}x_2 & + & a_{13}x_3 & = & b_1 \\ (R_2) & a_{21}x_1 & + & a_{22}x_2 & + & a_{23}x_3 & = & b_2 \\ (R_3) & a_{31}x_1 & + & a_{32}x_2 & + & a_{33}x_3 & = & b_3 \end{array}\right\} \Leftrightarrow \left[\begin{array}{ccc|c} a_{11} & a_{12} & a_{13} & b_1 \\ a_{21} & a_{22} & a_{23} & b_2 \\ a_{31} & a_{32} & a_{33} & b_3 \end{array}\right]$$

Example 1.2. Write the following linear system in matrix-vector form.

$$\begin{array}{rcrcrcl} x_1 & + & x_2 & + & x_3 & = & 1 \\ x_1 & + & 2x_2 & + & 3x_3 & = & 2 \\ 4x_1 & + & 5x_2 & + & 6x_3 & = & 1 \end{array}$$

2 Transforming system of linear equations: Elementary row operations

Elementary row operations are the following three rules for manipulating or transforming an augmented matrix that leave the values of the solution set unchanged.

Elementary row operations

1. Any two rows can be exchanged.
2. Any row may be multiplied (or divided) by a non-zero constant.
3. A multiple of any row can be added to any other row.

Elementary row operations are used in the **Gaussian elimination method** to transform the linear systems into the "upper triangular form", which can be used to obtain the solution easily.
Example: The following linear systems have the same solution $\{x_1, x_2, x_3\}$.

$$\text{(i)} \left\{ \begin{array}{lrcrcrcl} (R_1): & x_1 & + & x_2 & + & x_3 & = & 1 \\ (R_2): & x_1 & + & 2x_2 & + & 3x_3 & = & 2 \\ (R_3): & 4x_1 & + & 5x_2 & + & 6x_3 & = & 1 \end{array} \right\}$$

$$\text{(ii)} \left\{ \begin{array}{lrcrcrcl} (\bar{R}_1): & 2x_1 & + & 2x_2 & + & 2x_3 & = & 2 \\ (\bar{R}_2): & 0.5x_1 & + & x_2 & + & 1.5x_3 & = & 1 \\ (\bar{R}_3): & 4x_1 & + & 5x_2 & + & 6x_3 & = & 1 \end{array} \right\} (\bar{R}_1) = 2(R_1), (\bar{R}_2) = \frac{1}{2}(R_2)$$

$$\text{(iii)} \left\{ \begin{array}{lrcrcrcl} (\widehat{R}_1): & x_1 & + & x_2 & + & x_3 & = & 1 \\ (\widehat{R}_2): & & & x_2 & + & 2x_3 & = & 1 \\ (\widehat{R}_3): & 6x_1 & + & 7x_2 & + & 8x_3 & = & 3 \end{array} \right\} (\widehat{R}_2) = (R_2) - (R_1), (\widehat{R}_3) = (R_3) + 2(R_1)$$

3 Gaussian Elimination Method

Gaussian elimination method is a systematic way for solving a linear system. This method (with some modification) is generally well suited for solving *large-scale* linear systems on a computer. In particular, it is used for solving linear systems in the form:

$$\mathbf{A}\mathbf{x} = \mathbf{b}$$

by performing two procedures:

I. **Forward elimination**: Use **elementary row operations** to transform the system into the *upper triangular form*:

$$[\mathbf{A} \quad |\mathbf{b}] \quad \Rightarrow \quad [\tilde{\mathbf{A}} \quad |\tilde{\mathbf{b}}]$$

$$\left[\begin{array}{ccccc|c} a_{11} & a_{12} & a_{13} & \dots & a_{1n} & b_1 \\ a_{21} & a_{22} & a_{23} & \dots & a_{2n} & b_2 \\ a_{31} & a_{23} & a_{33} & \dots & a_{3n} & b_3 \\ \vdots & & & \ddots & \vdots & \vdots \\ a_{n1} & a_{n2} & a_{n3} & & a_{nn} & b_n \end{array}\right] \Rightarrow \left[\begin{array}{ccccc|c} \tilde{a}_{11} & \tilde{a}_{12} & \tilde{a}_{13} & \dots & \tilde{a}_{1n} & \tilde{b}_1 \\ 0 & \tilde{a}_{22} & \tilde{a}_{23} & \dots & \tilde{a}_{2n} & \tilde{b}_2 \\ 0 & 0 & \tilde{a}_{33} & \dots & \tilde{a}_{3n} & \tilde{b}_3 \\ \vdots & \vdots & & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & \tilde{a}_{nn} & \tilde{b}_n \end{array}\right]$$

II. **Back substitution**: To solve for the unknowns $\mathbf{x} = [x_1, \dots, x_n]^T$

- $x_n = \frac{b_n}{a_{nn}}$
- $\boxed{x_i = \frac{1}{a_{ii}}\left(b_i - \sum_{k=i+1}^{n} a_{ik}x_k\right)}$ for $i = n-1, n-2, \dots, 2, 1.$

Process of Forward Elimination in Gaussian Elimination Method Given a system of n linear equations with n unknowns.

- Write the augmented matrix of the system.
- Start from the top entry ("pivot entry") of the first nonzero column of this matrix

  Step 1: If the pivot entry is zero, swap the pivot row with some row below (so that the pivot entry is nonzero).

  Step 2: Eliminate all other entries below the pivot entry in the same column (i.e. make the entries below the pivot entry become zero) by subtracting suitable multiples of the pivot row from the other rows below.

  Step 3: Move the pivot entry down one row and over one column (to the right).

  - If all entries below the new pivot entry are zero, move to the next column.

Repeat **Step 1** until the process is done with column $n-1$ of the augmented matrix.

Example 3.1. Solve the following system of linear equations by using Gaussian elimination method.

$$\begin{aligned} x - y &= 2 \\ 2x - y - z &= 3 \\ x + y + z &= 6 \end{aligned}$$

Answer: The corresponding augmented matrix of the given linear system can be written as

$$\begin{array}{l} R_1 : \\ R_2 : \\ R_3 : \end{array} \quad \left[\begin{array}{rrr|r} 1 & -1 & 0 & 2 \\ 2 & -1 & -1 & 3 \\ 1 & 1 & 1 & 6 \end{array}\right] \Leftrightarrow \begin{array}{rcrcrcl} x & - & y & & & = & 2 \\ 2x & - & y & - & z & = & 3 \\ x & + & y & + & z & = & 6 \end{array}$$

and we will apply the following 2 procedures in Gaussian elimination to the augmented matrix.

I. Forward elimination to obtain the upper triangular form

II. Back substitution

I. Forward elimination

$$\begin{array}{l} R_1 : \\ R_2 : \\ R_3 : \end{array} \quad \left[\begin{array}{rrr|r} 1 & -1 & 0 & 2 \\ 2 & -1 & -1 & 3 \\ 1 & 1 & 1 & 6 \end{array}\right] \Leftrightarrow \begin{array}{rcrcrcl} x & - & y & & & = & 2 \\ 2x & - & y & - & z & = & 3 \\ x & + & y & + & z & = & 6 \end{array}$$

Step 1: R_1 is the "*pivot row*" and $a_{11} = 1$ is the "*pivot element*".
Let $m_{21} = \frac{a_{21}}{a_{11}} = \frac{2}{1} = 2$ and $m_{31} = \frac{a_{31}}{a_{11}} = \frac{1}{1} = 1$.

$$\begin{array}{l} R_1 : \\ R_2 \mapsto R_2 - m_{21}R_1 : \\ R_3 \mapsto R_3 - m_{31}R_1 : \end{array} \quad \left[\begin{array}{rrr|r} 1 & -1 & 0 & 2 \\ 0 & 1 & -1 & -1 \\ 0 & 2 & 1 & 4 \end{array}\right] \Leftrightarrow \begin{array}{rcrcrcr} x & - & y & & & = & 2 \\ & & y & - & z & = & -1 \\ & & 2y & + & z & = & 4 \end{array}$$

Step 2: R_2 is the "*pivot row*" and $a_{22} = 1$ is the "*pivot element*".
Let $m_{32} = \frac{a_{32}}{a_{22}} = \frac{2}{1} = 2$.

$$\begin{array}{l} R_1 : \\ R_2 : \\ R_3 \mapsto R_3 - m_{32}R_2 : \end{array} \quad \left[\begin{array}{rrr|r} 1 & -1 & 0 & 2 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 3 & 6 \end{array}\right] \Leftrightarrow \begin{array}{rcrcrcr} x & - & y & & & = & 2 \\ & & y & - & z & = & -1 \\ & & & & 3z & = & 6 \end{array}$$

II. Back substitution

- Row $R_3 : z = \frac{6}{3} = 2$
- Row $R_2 : y = -1 + z = -1 + 2 = 1$
- Row $R_1 : x = 2 + y = 3$

That is, the solution: $x = 3$, $y = 1$, $z = 2$. ■

Example 3.2. Solve the following system of linear equations by using Gaussian elimination method.

$$\begin{array}{rcrcrcl} 2x & + & 8y & + & 4z & = & 2 \\ 2x & + & 5y & + & z & = & 5 \\ 4x & + & 10y & - & z & = & 1 \end{array}$$

Example 3.3 (No solution)**.** Solve the following system of linear equations by using Gaussian elimination method.

$$\begin{array}{rcrcrcr} x & + & 2y & + & 3z & = & 0 \\ 4x & + & 5y & + & 6z & = & 3 \\ 7x & + & 8y & + & 9z & = & 0 \end{array}$$

Example 3.4 (Many solutions)**.** Solve the following system of linear equations by using Gaussian elimination method.

$$\begin{array}{ccccccc} 2x & + & 4y & + & 6z & = & 0 \\ 4x & + & 5y & + & 6z & = & 3 \\ 7x & + & 8y & + & 9z & = & 6 \end{array}$$