Sketch of solutions to some selected questions in Williamson* (*use this solution at your own risk!*)

Note: In the textbook and lecture, I use P(Q) to refer to the expected pay-off from job search. In this solution, the pay-off is denoted by *v(Q)*.

Search model

Problems

1. What does the DMP model predict would be the effects of labor-saving devices in the home, for example dishwashers, washing machines, and vacuum cleaners? Use diagrams to show the effects on the unemployment rate, the vacancy rate, the labor force, the number of firms, aggregate output, and labor market tightness, and discuss your results.
2. Suppose the government's goal is to reduce the unemployment rate. Some legislators propose that the government should give a subsidy s to any firm that hires a worker. Some other legislators argue that it would be more effective to simply pay consumers to stay home rather than searching for work, that is, anyone who chooses not to participate in the labor force should receive a payment q. Which policy is more effective in achieving the government's goal? Explain using the DMP model, with the aid of diagrams. [In your answer, do not concern yourself with how the subsidies from the government are financed.]
3. Suppose that there is technological change that reduces the cost of recruiting for firms. Using the DMP model, determine the effects on the unemployment rate, the vacancy rate, the labor force, the number of firms, aggregate output, and labor market tightness. Use diagrams, and explain your results.
4. Adapt the DMP model to include government activity as follows. Suppose that the government can operate firms, subject to the same constraints as private firms. In particular, the government must incur a cost k to post a vacancy. Supposing that the government operates G firms, then the number of matches in the economy as a whole is $M = em(Q, A + G)$, where A is the number of private firms that choose to post vacancies. Assume that the government pays the same wages as do private sector firms. Determine the effects of G on the unemployment rate, the vacancy rate, the labor force, the number of private firms, the total number of firms (private and government-run), aggregate output, and labor market tightness. Explain your results.
5. Show that, in the Keynesian DMP model, if the wage is judged to be inefficiently high, so that unemployment is inefficiently high, the government can pay a subsidy to firms that corrects the problem. Explain your results. Does it matter whether the government subsidizes firms that post vacancies or only successful matches? Discuss.
6. Suppose that all social programs simultaneously become more generous. In particular suppose that there is an increase in UI benefits, and also an increase in welfare benefits, which are represented in the DMP model as payments to everyone who is not in the labor force. What will be the effects on the unemployment rate, the vacancy rate, the labor force, the number of firms, the aggregate output, and the labor market tightness? Explain your results

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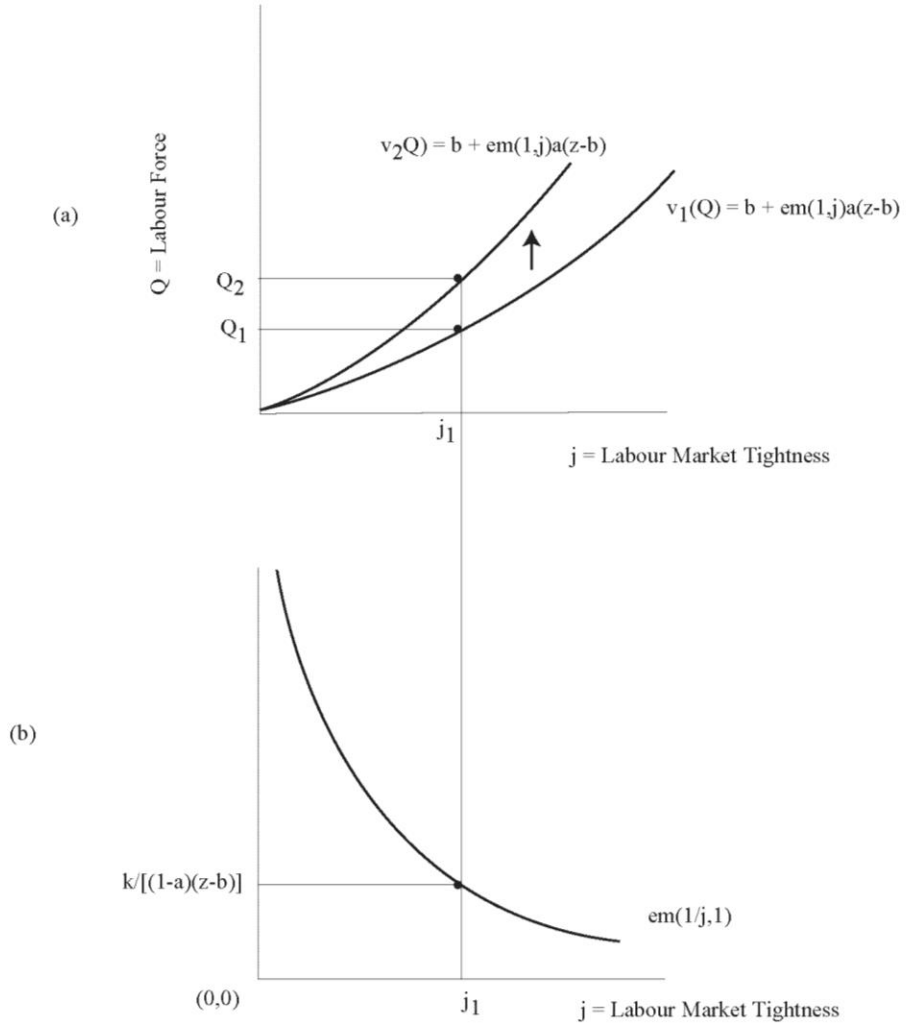
Question 1 More labor-saving devices has the effect of reducing the payoff to working at home for all consumers, which reduces *v(Q)* for each value of *Q*. As a result, the curve in panel (a) of Figure 6.1 shifts up. In equilibrium, *Q* increases, but *j* remains unchanged. The unemployment rate and the vacancy rate are unaffected, but the labor force *Q* increases.

Since $j = A/Q$, therefore the number of firms A increases. Aggregate output $Y = Qem(1,j)$, so Y increases, as Q has risen and j is unchanged. Labor saving devices makes searching for work more attractive relative to working at home for consumers. With more consumers in the market, labor market tightness tends to go down, which attracts more firms into the labour market. Ultimately, the number of active firms increases proportionally to the number of consumers searching for work, and there is no change in labor market tightness in equilibrium. Output goes up because there are more successful matches in the labor market.



Question 6 If all social welfare programs simultaneously become more generous, suppose that we represent this as a payment p to each person not in the labor force, and an increase by p in the employment insurance benefit. Then, the equation that summarizes behavior on the supply side of the labor market becomes

$$v(Q) + p = b + p + em(1,j)a(z-b-p),$$

or, simplifying,

$$vQ) = b + em(1,j)a(z-b-p).$$

As well, the equation summarizing demand-side behavior in the labor market can be written as

$$em(1/j,1) = k/(1-a)(z-b-p)$$

Therefore, in Figure 6.4, labor market tightness falls from j_1 to j_2, and the labor force falls from Q_1 to Q_2. As a result, the unemployment rate increases and the vacancy rate decreases. The number of firms is $A=jQ$, so A decreases. As well, output is $Y=zQem(1,j)$, so output falls as well. Consumers are affected by two social programs – one which pays a benefit to people not in the labor force, and one that pays an employment insurance benefit to the unemployed. Since the consumer receives the employment insurance benefit only in the event that search for work is unsuccessful, the increase in generosity of all social programs will on net discourage consumers from searching for work. Further, more generous social programs reduces the total surplus from a successful match, and this discourages firms from posting vacancies. On net, labor market tightness goes down, the labor force contracts, and aggregate

output decreases, with the unemployment rate increasing and the vacancy rate decreasing.

Solow model

Problems

1. In the Malthusian model, suppose that the quantity of land increases. Using diagrams, determine what effects this has in the long-run steady state and explain your results.
2. In the Malthusian model, suppose that there is a technological advance that reduces death rates. Using diagrams, determine the effects of this in the long-run steady state and explain your results.
3. In the Solow growth model, suppose that the marginal product of capital increases for each quantity of the capital input, given the labor input.
   (a) Show the effects of this on the aggregate production function.
   (b) Using a diagram, determine the effects on the quantity of capital per worker and on output per worker in the steady state.
   (c) Explain your results.
4. Suppose that the depreciation rate increases. In the Solow growth model, determine the effects of this on the quantity of capital per worker and on output per worker in the steady state. Explain the economic intuition behind your results.
5. Suppose that the economy is initially in a steady state and that some of the nation's capital stock is destroyed because of a natural disaster or a war.
   (a) Determine the long-run effects of this on the quantity of capital per worker and on output per worker.
   (b) In the short run, does aggregate output grow at a rate higher or lower than the growth rate of the labor force?
   (c) After World War II, growth in real GDP in Germany and Japan was very high. How do your results in parts (a) and (b) shed light on this historical experience?
6. If total factor productivity decreases, determine using diagrams how this affects the golden rule quantity of capital per worker and the golden rule savings rate. Explain your results.
7. Modify the Solow growth model by including government spending as follows. The government purchases G units of consumption goods in the current period, where $G = gN$ and g is a positive constant. The government finances its purchases through lump-sum taxes on consumers, where T denotes total taxes, and the government budget is balanced each period, so that $G = T$. Consumers consume a constant fraction of disposable income—that is, $C = (1 - s)(Y - T)$, where s is the savings rate, with $0 < s < 1$.
   (a) Derive equations similar to Equations (7-18), (7-19), and (7-20), and show in a diagram how the quantity of capital per worker, k^*, is determined.
   (b) Show that there can be two steady states, one with high k^* and the other with low k^*.
   (c) Ignore the steady state with low k^* (it can be shown that this steady state is "unstable"). Determine the effects of an increase in g on capital per worker and on output per worker in the steady state. What are the effects on the growth rates of aggregate output, aggregate consumption, and aggregate investment?
   (d) Explain your results.
8. Determine the effects of a decrease in the population growth rate on the golden rule quantity of capital per worker and on the golden rule savings rate. Explain your results.
9. Consider a numerical example using the Solow growth model. Suppose that $F(K, N) = K^{0.5}N^{0.5}$, with $d = 0.1$, $s = 0.2$, $n = 0.01$, and $z = 1$, and take a period to be a year.
   (a) Determine capital per worker, income per capita, and consumption per capita in the steady state.

(b) Now, suppose that the economy is initially in the steady state that you calculated in part (a). Then, s increases to 0.4.
   i. Determine capital per worker, income per capita, and consumption per capita in each of the 10 years following the increase in the savings rate.
   ii. Determine capital per worker, income per capita, and consumption per capita in the new steady state.
   iii. Discuss your results; in particular comment on the speed of adjustment to the new steady state after the change in the savings rate, and the paths followed by capital per worker, income per capita, and consumption per capita.

10. Suppose that we modify the Solow growth model by allowing long-run technological progress. That is, suppose that $z = 1$ for convenience, and that there is labor-augmenting technological progress, with a production function

$$Y = F(K, bN),$$

where b denotes the number of units of "human capital" per worker, and bN is "efficiency units" of labor. Letting b' denote future human capital per worker, assume that $b' = (1+f)b$, where f is the growth rate in human capital.
   (a) Show that the long-run equilibrium has the property that $k^{**} = \frac{K}{bN}$ is a constant. At what rate does aggregate output, aggregate consumption, aggregate investment, and per capita income grow in this steady state? Explain.
   (b) What is the effect of an increase in f on the growth in per capita income? Discuss relative to how the standard Solow growth model behaves.

11. Alter the Solow growth model so that the production technology is given by $Y = zK$, where Y is output, K is capital, and z is total factor productivity. Thus, output is produced only with capital.
   (a) Show that it is possible for income per person to grow indefinitely.
   (b) Also show that an increase in the savings rate increases the growth rate in per capita income.
   (c) From parts (a) and (b), what are the differences between this model and the basic Solow growth model? Account for these differences and discuss.

12. Consider a numerical example. In the Solow model, assume that $n = 0$, $s = 0.2$, $d = 0.1$, and $F(K, N) = K^{0.3}N^{0.7}$. Suppose that initially, in period $t = 0$, $z = 1$ and the economy is in a steady state.
   (a) Determine consumption, investment, savings, and aggregate output in the initial steady state.
   (b) Suppose that at $t = 1$, total factor productivity falls to $z = 0.9$ and then returns to $z = 1$ for periods $t = 2, 3, 4, \ldots$. Calculate consumption, investment, savings, and aggregate output for each period $t = 1, 2, 3, 4, \ldots$.
   (c) Repeat part (b) for the case where, at $t = 1$, total factor productivity falls to $z = 0.9$ and then stays there forever.
   (d) Discuss your results in parts (a)–(c).

13. Consider the following data:

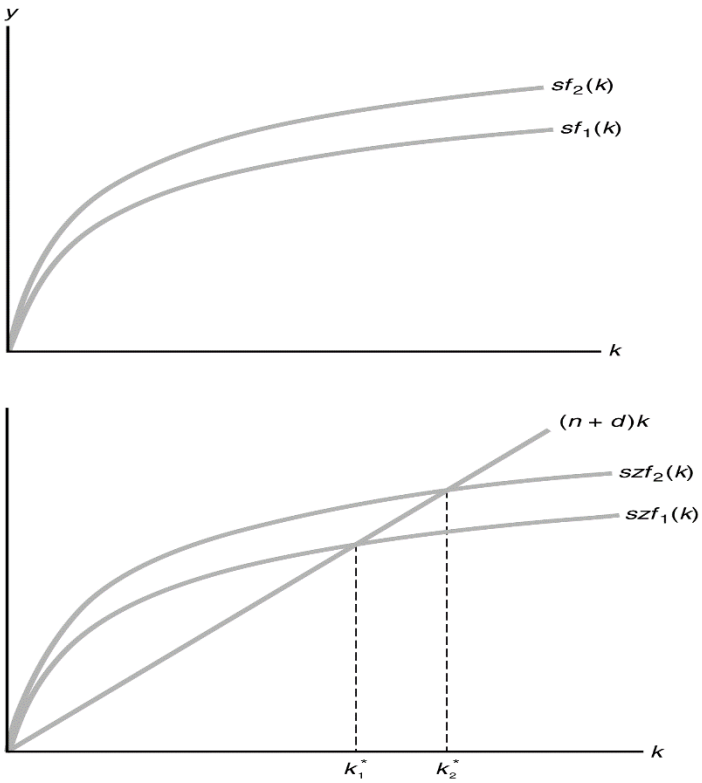
Year	$\hat{Y}$ (billions of 2005 dollars)	$\hat{K}$ (billions of 2005 dollars)	$\hat{N}$ (millions)
1995	9086.0	31438.0	124.9
1996	9425.8	32338.4	126.7
1997	9845.9	33307.7	129.6
1998	10274.7	34428.0	131.5
1999	10770.7	35679.0	133.5
2000	11216.4	36999.0	136.9
2001	11337.5	38164.0	136.9
2002	11543.1	39233.9	136.5
2003	11836.4	40322.6	137.7
2004	12246.9	41471.4	139.2
2005	12623.0	42609.9	141.7
2006	12958.5	43836.6	144.4
2007	13206.4	44949.2	146.1

1. (a) Calculate the Solow residual for each year from 1995 to 2007.

(b) Calculate percentage rates of growth in output, capital, employment, and total factor productivity for the years 1996 to 2007. In each year, what contributes the most to growth in aggregate output? What contributes the least? Are there any surprises here? If so, explain.

Question 3 For the marginal product of capital to increase at every level of capital, the shift in the production function is equivalent to an increase in total factor productivity.

(a) The original and new production functions are depicted in the figures below.



(b) Equilibrium in the Solow model is at the intersection of $szf(k)$ with the line segment $(n+d)k$. The old and new equilibria are depicted in the bottom panel of the figure above. The new equilibrium is at a higher level of capital per capita and a higher level of output per capita.

(c) For a given savings rate, more effective capital implies more savings, and in the steady state there is more capital and more output. However, if the increase in the marginal product of capital were local, in the neighborhood of the original equilibrium, there would be no equilibrium effects. A twisting of the production function around its initial point does not alter the intersection point.

Question 5. A destruction of capital.

(a) The long-run equilibrium is not changed by an alteration of the initial conditions. If the economy started in a steady state, the economy will return to the same steady state. If the economy were initially below the steady state, the approach to the steady state will be delayed by the loss of capital.

(b) Initially, the growth rate of the capital stock will exceed the growth rate of the labor force. The faster growth rate in capital continues until the steady state is reached.

(c) The rapid growth rates are consistent with the Solow model's predictions about the likely adjustment to a loss of capital.

Question 7. Government spending in the Solow model.

(a) By assumption, we know that T = G, and so we may write:

$$K' = s(Y - G) + (1 - d)K = sY - gN + (1 - d)K$$

Now divide by *N* and rearrange as:

$$k'(1 + n) = szf(k) - sg + (1 - d)k$$

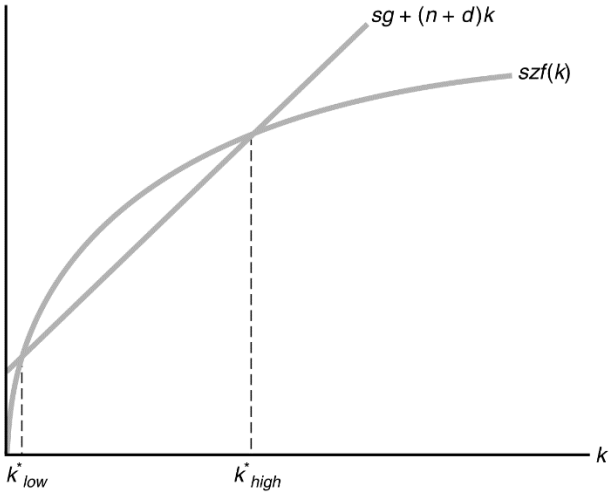
Divide by (1 + *n*) to obtain:

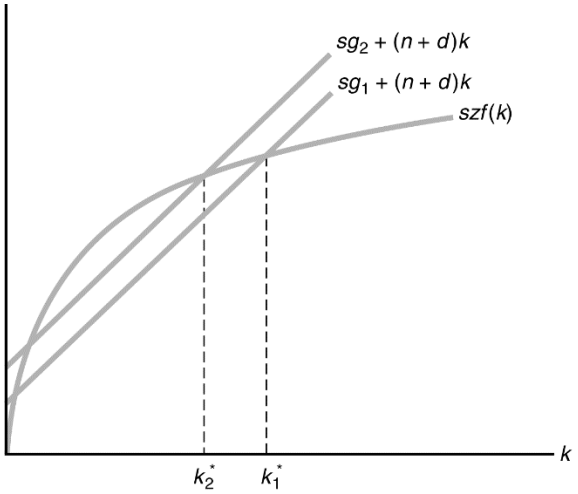
$$k' = \frac{szf(k)}{(1 + n)} - \frac{sg}{(1 + n)} + \frac{(1 - d)k}{(1 + n)}$$

Setting $k = k'$, we find that:

$$szf(k^*) = sg + (n + d)k^*.$$

This equilibrium condition is depicted in the figure below.





(b) The two steady states are also depicted in the figure above.

(c) The effects of an increase in *g* are depicted in the bottom panel of the figure above. Capital per capita declines in the steady state. Steady-state growth rates of aggregate output, aggregate consumption, and investment are all unchanged. The reduction in capital per capita is accomplished through a temporary reduction in the growth rate of capital.

Question 8. The golden rule quantity of capital per capita, k^*, is such that $MP_K = zf'(k^*) = n + d$. A decrease in the population growth rate, *n*, requires a decrease in the marginal product of capital. Therefore, the golden rule quantity of capital per capita must increase. The golden rule savings rate may either increase or decrease. **(Why? Didn't I say in class that there is a one-to-one relationship between "s" and kg*? Yes, I said that. But why the solution conclude this way. Think!)**

9. a) Given the production function, we can write the per-worker production function as $zf(k) = zk^{0.5}$

Then, from Equation 6.19 the steady state quantity of capital per worker, *k*, is determined by $0.2k^{0.5} = 0.11k,$

so solving for *k* we get *k* = 3.3058. Then, income per capita is $(3.3058)^{0.5}$ = 1.8182. Finally, consumption per capita is given by 1.8182(1-*s*) = 1.4546.

b)

Period	*k*	*y*	*c*
1	3.96	1.99	1.19
2	4.67	2.16	1.30
3	5.43	2.33	1.40
4	6.25	2.50	1.50
5	7.11	2.67	1.60
6	8.02	2.83	1.70
7	8.98	3.00	1.80
8	9.99	3.16	1.90
9	11.04	3.32	1.99
10	12.14	3.48	2.09

In the new steady state, with *s* = 0.4, calculating the steady state as before, we get *k* = 13.22, *y* = 3.64, and *c* = 2.18. Note that after 10 periods, the economy is much closer to the new steady state than to the old steady state with the lower savings rate. Of particular interest is the fact that consumption per capita actually decreases initially relative to the initial steady state, but consumption per person will actually be higher in the new steady state than in the initial one. This effect occurs because, with a higher saving rate, consumption must initially fall, but as the capital stock rises, the higher level of output tends to increase consumption.

Endogenous growth model

1. Could differences across countries in population growth account for the persistence in income disparity across countries? Use the Solow growth model to address this question and discuss.

2. In the Solow growth model, suppose that the per-worker production function is given by $y = zk^{.3}$, with $s = 0.25$, $d = 0.1$, and $n = 0.02$.
   (a) Suppose that in country A, $z = 1$. Calculate per capita income and capital per worker.
   (b) Suppose that in country B, $z = 2$. Calculate per capita income and capital per worker.
   (c) As measured by GDP per capita, how much richer is country B than country A? What does this tell us about the potential for differences in total factor productivity to explain differences in standards of living across countries?

3. Suppose that there are two countries with different levels of total factor productivity, and that these differences exist because of barriers to technology adoption in the low-productivity country. Also suppose that these two countries do not trade with each other. Now, suppose that residents of each country were free to live in either country. What would happen, and what conclusions do you draw from this?

4. Suppose, in the Solow growth model, that learning by doing is captured as a cost of installing new capital. In particular, suppose that for each unit of investment, r units of goods are used up as a cost to firms.
   (a) Determine how r affects the steady state quantity of capita per worker, and per capita income.
   (b) Now suppose that r differs across countries. How will these countries differ in the long run? Discuss.

5. Suppose that z, the marginal product of efficiency units of labor, increases in the endogenous growth model. What effects does this have on the rates of growth and the levels of human capital, consumption, and output? Explain your results.

6. Introduce government activity in the endogenous growth model as follows. In addition to working u units of time in producing goods, the representative consumer works v units of time for the government and produces gvH goods for government use in the current period, where $g > 0$. The consumer now spends $1 - u - v$ units of time each period accumulating human capital.
   (a) Suppose that v increases with u decreasing by an equal amount. Determine the effects on the level and the rate of growth of consumption. Draw a diagram showing the initial path followed by the natural logarithm of consumption and the corresponding path after v increases.
   (b) Suppose that v increases with u held constant. Determine the effects on the level and the rate of growth of consumption. Draw a diagram showing the initial path followed by the natural logarithm of consumption and the corresponding path after v increases.
   (c) Explain your results and any differences between parts (a) and (b).

7. Suppose that the government makes a one-time investment in new public school buildings,

which results in a one-time reduction in consumption. The new public school buildings increase the efficiency with which human capital is accumulated. Determine the effects of this on the paths of aggregate consumption and aggregate output over time. Is it clear that this investment in new schools is a good idea? Explain.

8. Reinterpret the endogenous growth model in this chapter as follows. Suppose that there are two groups of people in a country, the low-skilled workers and the high-skilled workers. The low-skilled workers have less human capital per person initially than do the high-skilled workers. In the economy as a whole, output is produced using efficiency units of labor, and total factor productivity is z, just as in the endogenous growth model in this chapter. Each individual in this economy accumulates human capital on their own, and each has one unit of time to split between human capital accumulation and work. However, now $b = b_h$ for the high-skilled, $b = b_l$ for the low-skilled, $u = u_h$ for the high-skilled, and $u = u_l$ for the low-skilled. In the United States, there has been an increase in the gap between the wages of high-skilled workers and low-skilled workers, that has occurred over the last 30 years or so. Determine how this model can explain this observation, and discuss.

9. Suppose there are two countries. In the rich country, the representative consumer has H_r units of human capital, and total factor productivity is z_r. In the poor country, the representative consumer has H_p units of human capital, and total factor productivity is z_p. Assume that b and u are the same in both the countries, $H_r > H_p$, and $z_r > z_p$.
   (a) How do the levels of per capita income, the growth rates of per capita income, and real wages compare between the rich and poor countries?
   (b) If consumers could choose their country of residence, where would they want to live?
   (c) If each country could determine immigration policy, what should they do to maximize the welfare of the current residents?
   (d) What is the immigration policy that maximizes the welfare of the citizens of both countries?
   (e) Explain your results. Do you think this is a good model for analyzing the effects of immigration? Why or why not?

10. In the endogenous growth model, suppose that there are three possible uses of time. Let u denote the fraction of time spent working, s the fraction of time spent neither working nor accumulating human capital (call this *unemployment*), and $1 - u - s$ the fraction of time spent accumulating human capital. Assume that $z = 1$ and $b = 4.2$. Also assume that the economy begins period 1 with 100 units of human capital.
   (a) Suppose that for periods 1, 2, 3, . . . , 10, $u = .7$ and $s = 0.05$. Calculate aggregate consumption, output, and the quantity of human capital in each of these periods.
   (b) Suppose that, in period 11, $u = 0.6$ and $s = 0.15$. Then, in periods 12, 13, 14, . . . , $u = 0.7$ and $s = 0.05$. Calculate aggregate consumption, output, and the quantity of human capital in periods 11, 12, 13, . . . , 20.
   (c) Suppose alternatively that in period 11, $u = 0.6$ and $s = 0.05$. Again, calculate aggregate consumption, output, and the quantity of human capital in periods 11, 12, 13, . . . , 20.
   (d) Now suppose that in period 11, $u = 0.6$ and $s = 0.10$. Calculate aggregate consumption, output, and the quantity of human capital in periods 11, 12, 13, . . . , 20.
   (e) What do you conclude from your results in parts (a)–(d)? Discuss.

1. Differences in population growth rates may account for differences in the equilibrium levels of capital per capita and output per capita across otherwise identical economies. However, once we have isolated this difference, countries well below their equilibrium growth paths should be growing more rapidly than countries that are closer to their equilibrium growth paths. That is, we should observe convergence.

2. We want to solve Equation 6.19 for the given parameter values: $0.25\, zk^3 = (0.01 + 0.02)\, k$. From this we obtain $k = (0.25/0.03)^{(1/0.7)} z^{(1/0.7)} = 20.675 z^{1.43}$. Then $y = zk^3$.

   (a) If $z = 1$, then $k = 20.675$ and $y = 2.48$.
   (b) If $z = 2$, then $k = 55.708$ and $y = 6.48$.
   (c) Country B is producing 160% more output than country A, despite having only 100% more total factor productivity, because differences in z have been amplified by capital accumulation. Differences in TFP are thus an interesting avenue to explain differences across countries.

3. From Figure 7.20 in Chapter 7, per capita capital and per capita income will be higher in the country with higher TFP. From Chapter 4, higher TFP implies that the marginal product of labor will be higher, for given capital and labor inputs. As well, from Chapter 4 higher capital implies that the marginal product of labor is higher. Therefore, in the country with high TFP, the marginal product of capital will be higher in the steady state than in the country with low TFP. Thus, if workers migrate to the country with higher wages, they will choose to migrate from the low-TFP country to the high-TFP country until wages are equalized. In the world, the high-income countries tend to be those with high TFP, and immigration tends to flow from less-developed to more-developed countries. Indeed, high-income countries tend to erect immigration barriers, which prevent wages from equalizing across countries.

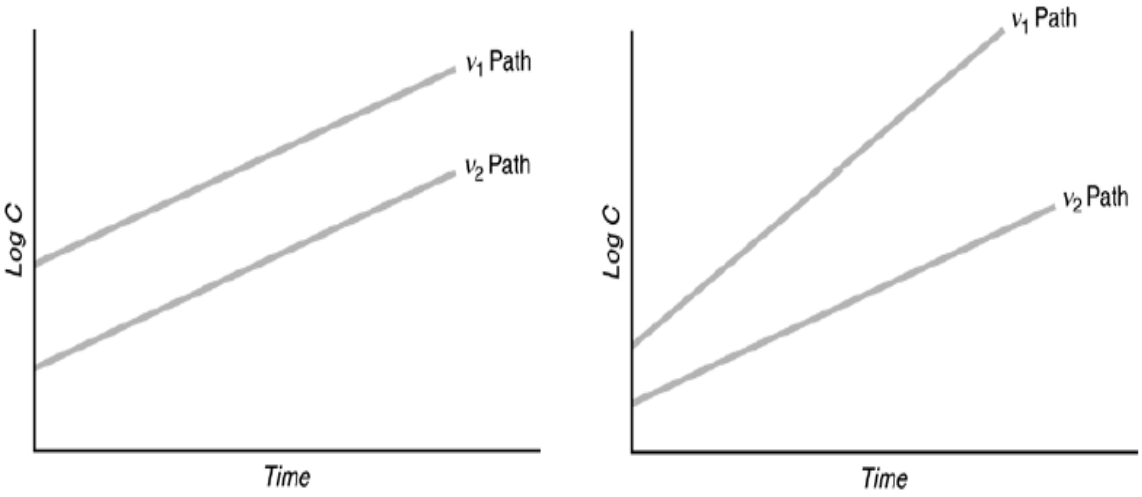
5. An increase in the marginal product of efficiency unit of labor increases the real wage rate, and increases output. However, the increase in z does not change the equilibrium growth rates. The economy has higher paths for consumption and output, but the two paths share the same growth rate.

6. Government activity in the endogenous growth model.

   (a) The equation of motion for the economy is now given by:

$$H' = b(1-u-v)H$$

A change in v, holding $u+v$ constant, has no effect on the path of H. Consumption is lower because the time spent working for the government cannot produce consumable goods. The two paths of $\log C$ are depicted in the left figure below.



(b) Holding u constant, an increase in v reduces the growth rate of human capital. The level of consumption falls as workers are taken away from producing consumption goods. The growth rate of consumption also decreases due to the reduction of the growth rate of human capital. The two consumption paths are depicted in the right figure above.

(c) Offsetting changes in u and v change the level of consumption. However, the equation of motion for H is unchanged, so the rate of growth is unchanged. In part c, the growth rate of H is changed, and so is the growth rate of C.

8. As there is no interaction between the two groups, we can study them individually and then compare them. There are four relevant cases to analyze:

Case 1: $b_h > b_l$ and $u_h > u_l$. Here, high-skilled people spend less time at school, but are more efficient at accumulating human capital while there. Whether their human capital, and thus their wage, grows faster is not clear, it depends whether $b_h(1 - u_h)$ is larger than $b_l(1 - u_l)$ or not.

Case 2: $b_h > b_l$ and $u_h < u_l$. This means high-skilled people are learning more efficiently and are spending more time at school. Their human capital thus grows faster, therefore their wage grows faster. This can explain the growing wage gap in the United States between high-skilled and low-skilled workers.

Case 3: $b_h < b_l$ and $u_h > u_l$. This is the exact opposite of case 2. Here low-skilled people would eventually overtake high-skilled people and would command a higher wage. This is clearly not what is happening in the United States.

Case 4: $b_h < b_l$ and $u_h < u_l$. This case is similar to case 1 in the sense that the impact on human capital differences depends on whether $b_h(1 - u_h)$ is larger than $b_l(1 - u_l)$ or not.