

EE320 Lecture Note
Chapter 6: Optimization without Constraints:
One Independent Variable Case

1 Introduction

Optimal value in economics

- maximizing some economic variable i.e. profit.
- minimizing some economic variable i.e. cost.

by looking at the objective function - dependent variable
and making choice - independent variable.

ex. Firm's problem:

$$\Pi(Q) \equiv R(Q) - C(Q)$$

where Π is the object of maximization.
 Q is the choice variable.

2 Maxima, Minima and Inflection point

2.1 Necessary Conditions

Consider this function $y = f(x)$

A, B = relative maxima
 D = relative minima
 C = absolute maxima
 E = absolute minima
 F = inflection point

To test that point, find 1st - order derivative

$$\left. \begin{array}{l} \frac{dy}{dx}|_A = 0 \\ \frac{dy}{dx}|_B = 0 \end{array} \right\} \text{relative max}
 \qquad
 \left. \begin{array}{l} \frac{dy}{dx}|_D = 0 \\ \frac{dy}{dx}|_E = 0 \end{array} \right\} \text{relative min}$$

$$\frac{dy}{dx}|_F = 0 \quad \} \text{inflection point}$$

These are “necessary condition” for relative maximum/minimum. (still can’t distinguish whether it is maximum, minimum or inflection point)

ex. $y = x^3 - 12x^2 + 36x + 8$

FOCs : $\frac{dy}{dx} = 3x^2 - 24x + 36 = 0$ find tangent line that slope is zero.

$$(x - 2)(x - 6) = 0$$

2.2 Sufficient Condition (2^{nd} order derivative)

Suppose $y = f(x)$

$$\frac{dy}{dx} = y' = f'(x) \quad 1^{st} \text{ order derivative}$$

$$\frac{d^2y}{dx^2} = y'' = f''(x) \quad 2^{nd} \text{ order derivative}$$

Note $\frac{d}{dx} \left(\frac{dy}{dx} \right) = \frac{d^2y}{dx^2}$

ex. $y = x^3$

$$\frac{dy}{dx} = 3x^2 \quad \text{slope of the tangent line} \Rightarrow \frac{dy}{dx} > 0 \text{ when } x \uparrow, y \uparrow.$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx} \left(\frac{dy}{dx} \right) = 6x \quad \text{change in the slope of tangent line when } x \text{ changes}$$

$$\Rightarrow \frac{d^2y}{dx^2} > 0 \text{ when } x \uparrow, \frac{dy}{dx} \uparrow$$

$$\frac{d^3y}{dx^3} = 6$$

For both $\frac{dy}{dx} > 0$ and $\frac{d^2y}{dx^2} > 0$ means the slope is positive and it is increasing.

\therefore Interpretation of 1^{st} and 2^{nd} derivative :

1^{st} derivative $f'(x) > 0$ the value of the function tends to increase.
 $f'(x) < 0$ the value of the function tends to decrease.

2^{nd} derivative $f''(x) > 0$ the slope of the function tends to increase.
 $f''(x) < 0$ the slope of the function tends to decrease.

Possible cases

- 1) $f'(x_0) > 0$ and $f''(x_0) > 0$ slope is positive and increasing as x increases.
- 2) $f'(x_0) < 0$ and $f''(x_0) > 0$ slope is negative and increasing as x increases.
- 3) $f'(x_0) > 0$ and $f''(x_0) < 0$ slope is positive and decreasing as x increases.
- 4) $f'(x_0) < 0$ and $f''(x_0) < 0$ slope is negative and decreasing as x increases.

Note inflection point: $\frac{dy}{dx} = 0$ and $\frac{d^2y}{dx^2} = 0$

3 Convexity and Concavity

Definition: strictly convex

A function $f(x)$ is strictly convex if we pick any pair of points M and N on its curve and join them by a straight line, the line segment MN must lie entirely above the curve, except at points M and N .

Definition: strictly concave

A function $f(x)$ is strictly concave if we pick any pair of points M and N on its curve and join them by a straight line, the line segment MN must lie entirely below the curve, except at points M and N .

Or assume that $f(x)$ is continuous and twice differentiable:

if $f''(x) > 0 \quad \forall X$, then $f(x)$ is strictly convex function.

if $f''(x) < 0 \quad \forall X$, then $f(x)$ is strictly concave function.

ex. 1.increasing convex

$$f'(x) > 0$$

$$f''(x) > 0$$

2. decreasing convex

$$f'(x) < 0$$

$$f''(x) > 0$$

3. increasing concave

$$f'(x) > 0$$

$$f''(x) < 0$$

4. decreasing concave

$$f'(x) < 0$$

$$f''(x) < 0$$

ex. Attitude towards risk.

risk-averse preference $U(x)$ is concave

risk-loving preference $U(x)$ is convex

Conclusion $y = f(x)$ To find the maximum or minimum.

FOC: $\frac{dy}{dx} = 0$ solve for critical value that its tangent line has slope = 0

SOC: $\frac{d^2y}{dx^2} > 0$ $f''(x_0) > 0$ minimum (slope increases)

$\frac{d^2y}{dx^2} < 0$ $f''(x_0) < 0$ maximum (slope decreases)

ex. Determine the extreme of $f(x) = x^3 - 3x^2 + 2$

$$f'(x) = 3x^2 - 6x = 0 \rightarrow 3x(x - 2) = 0$$

$$x^* = 0, 2$$

$$f''(x) = 6x - 6 = 0 \rightarrow \begin{aligned} f''(0) &= -6 < 0 \rightarrow f(0) = \max \\ f''(2) &= 6 > 0 \rightarrow f(2) = \min \end{aligned}$$

$$\underline{\text{ex.}} \quad g(x) = \frac{x^4}{4} - \frac{3}{2}x^2$$

$$g'(x) = x^3 - 3x = 0$$

$$x(x^2 - 3) = 0$$

$$x^* = 0, \sqrt{3}, -\sqrt{3}$$

$$\begin{aligned} g''(X) = 3x^2 - 3 \rightarrow g''(0) &= -3 < 0 \rightarrow g(0) &= \text{max} &= 0 \\ g''(\sqrt{3}) &= 6 > 0 \rightarrow g(\sqrt{3}) &= \text{min} &= -\frac{9}{4} \\ g''(-\sqrt{3}) &= 6 > 0 \rightarrow g(-\sqrt{3}) &= \text{min} &= -\frac{9}{4} \end{aligned}$$

$$\underline{\text{ex.}} \quad f(x) = x^4$$

$$f'(x) = 4x^3 = 0 \rightarrow x^* = 0 \quad \underline{\text{FOC}}$$

$$f''(x) = 12x^2 \rightarrow f''(0) = 0 \quad \text{but } \underline{\text{SOC}} \text{ does } \underline{\text{not}} \text{ work}$$

but we know $12x^2 \geq 0$

$$\underline{\text{ex.}} \quad f(x) = x^3$$

$$f'(x) = 3x^2 = 0 \rightarrow x^* = 0$$

$$f''(x) = 6x \rightarrow f''(0) = 0$$

* Therefore when we get $f''(x_0) = 0$, it could be

1. maximum or minimum ex. $y = x^4$ slope changes its sign.
2. inflection point ex. $y = x^3$ slope remains in the same sign.

conditions	maximum	minimum
1. first-order necessary	$f'(x) = 0$	$f'(x) = 0$
2. second-order necessary	$f''(x) \leq 0$	$f''(x) \geq 0$
3. second-order sufficient	$f''(x) < 0$	$f''(x) > 0$

4 Profit maximization problem

$$\Pi(Q) = TR(Q) - TC(Q)$$

$$\text{Max. } \Pi(Q) = TR(Q) - TC(Q)$$

$$\text{FOC: set } \frac{d\Pi(Q)}{dQ} = 0$$

$$\Rightarrow \frac{d}{dQ}\Pi(Q) = \frac{d}{dQ}TR(Q) - \frac{d}{dQ}TC(Q) = 0$$

$$MR(Q) - MC(Q) = 0$$

$$MR(Q) = MC(Q) \text{ profit-maximizing necessary condition}$$

For max profit, we want $\frac{d^2\Pi(Q)}{dQ^2} < 0$ (so that we have maximum value of $\Pi(Q)$)

$$\text{SOC: } \frac{d^2\Pi(Q)}{dQ^2} < 0$$

$$\Rightarrow \frac{d^2\Pi(Q)}{dQ^2} = \frac{d^2TR(Q)}{dQ^2} - \frac{d^2TC(Q)}{dQ^2} < 0$$

$$MR'(Q) - MC'(Q) < 0$$

$$MR'(Q) < MC'(Q) \quad \text{sufficient condition}$$

$$\text{slope of MR} < \text{slope of MC}$$

2 cases to consider $\begin{cases} \text{Perfect Competition} \\ \text{Monopoly} \end{cases}$

✓ Perfect Competition

Assumptions :

1. many buyers and sellers \Rightarrow firms as price taker
2. homogeneous product
3. no barrier to entry or exit etc.

$$TR(Q) = \bar{P} \cdot Q$$

$$TC(Q) = C(Q)$$

$$\Pi(Q) = TR(Q) - TC(Q)$$

$$FOC: \frac{d}{dQ}\Pi(Q) = \bar{P} - MC(Q) = 0$$

$$\bar{P} = MC(Q)$$

$$SOC : \begin{aligned} \frac{d^2}{dQ^2}\Pi(Q) &< 0 \\ -MC'(Q) &< 0 \\ MC'(Q) &> 0 \end{aligned}$$

ex. $P = 10$, $TC = Q^2$

Find Π_{max} , Q and show that Π is maximum.

$$\begin{aligned} \Rightarrow \Pi(Q) &= TR(Q) - TC(Q) \\ &= 10Q - Q^2 \end{aligned}$$

$$FOC : \begin{aligned} \frac{d\Pi}{dQ} &= 10 - 2Q = 0 \\ Q^* &= 5 \end{aligned}$$

$$SOC : \begin{aligned} \frac{d^2\Pi}{dQ^2} &= -2 < 0 \quad \text{max *SOSC is satisfied.} \\ \Pi(5) &= 10 \cdot (5) - 5^2 = 25 \quad \text{maximum profit} \end{aligned}$$

ex. $P = 30$, $TC = 100 + 19Q - 5Q^2 + \frac{1}{3}Q^3$

Find Q^* maximizing profit

$$\Rightarrow \Pi(Q) = 30Q - 100 - 19Q + 5Q^2 - \frac{1}{3}Q^3$$

$$FOC : \begin{aligned} \frac{d}{dQ}\Pi &= 30 - 19 + 10Q - Q^2 = 0 \\ Q^2 - 10Q - 11 &= 0 \\ (Q - 11)(Q + 1) &= 0 \\ Q^* &= -1, 11 \end{aligned}$$

$$\begin{aligned}
 SOC : \frac{d^2\Pi}{dQ^2} &= 10 - 2Q \\
 \Pi''(11) &= 10 - 22 = -12 < 0 \quad \text{max. *SOSC is satisfied.}
 \end{aligned}$$

$$\Pi(11) = 30(11) - 100 - 19(11) + 5(11)^2 - \frac{1}{3}(11)^3$$

✓ Monopoly

Producer/seller can set the price in the market.

$$TR(Q) = P(Q) \cdot Q$$

$$TC(Q) = C(Q)$$

$$\Pi(Q) = TR(Q) - C(Q)$$

$$\begin{aligned}
 FOC : \frac{d\Pi}{dQ} &= [P(Q) \cdot (1) + Q \cdot P'(Q)] - MC(Q) = 0 \\
 MR(Q) &= MC(Q)
 \end{aligned}$$

$$SOC: \frac{d^2}{dQ^2}\Pi(Q) < 0 \Rightarrow MR'(Q) < MC'(Q)$$

ex. $P = 4000 - 33Q$; $TC = 2Q^3 - 3Q^2 + 400Q - 5000$, find Q^*_{max} that maximize Π .

$$\Pi(Q) = \underbrace{(4000 - 33Q)Q}_{TR} - \underbrace{2Q^3 + 3Q^2 - 400Q + 5000}_{TC}$$

$$\begin{aligned}
 FOC : \frac{d\Pi}{dQ} &= 4000 - 66Q - 6Q^2 + 6Q - 400 = 0 \\
 &Q^2 + 10Q - 600 = 0 \\
 &(Q - 20)(Q + 30) = 0 \\
 &Q^* = 20 \text{ or } -30
 \end{aligned}$$

$$\begin{aligned}
 SOC : \frac{d^2\Pi}{dQ^2} &= -72 - 12Q \\
 \Pi''(20) &= -312 < 0 \quad \text{max.} \quad \text{*SOSC is satisfied.} \\
 \Pi''(30) &= 288 > 0 \quad \text{min.}
 \end{aligned}$$

ex. $P = 48 - 0.5Q$, $TC = 2 + 60Q - 8Q^2 + Q^3$. Find Q^*

$$\Pi(Q) = \underbrace{(48 - 0.5Q)Q}_{TR} - \underbrace{2 - 60Q + 8Q^2 - Q^3}_{TC}$$

$$\begin{aligned}
 FOC: \frac{d}{dQ}\Pi(Q) &= 48 - Q - 60 + 16Q - 3Q^2 = 0 \\
 &-3Q^2 + 15Q - 12 = 0 \\
 &Q^2 - 5Q + 4 = 0 \\
 &(Q - 4)(Q - 1) = 0 \\
 &Q^* = 1, 4
 \end{aligned}$$

$$SOC: \frac{d^2\Pi}{dQ^2} = -6Q + 15 = 0$$

$$\Pi''(1) = 9 > 0 \quad \text{min.}$$

$$\Pi''(4) = -9 < 0 \quad \text{max.}$$

$$\Pi(4) = TR(4) - TC(4) = 6$$

$$\Pi(1) = ?$$

Monopoly power

$$TR = P(Q) \cdot Q$$

$$\frac{dTR}{dQ} = P(Q) \frac{dQ}{dQ} + Q \frac{dP(Q)}{dQ}$$

$$MR = P(Q) + Q \frac{d}{dQ} P(Q) \cdot \frac{P(Q)}{P(Q)}$$

$$= P(Q) + \left[\frac{Q}{P(Q)} \cdot \frac{d}{dQ} P(Q) \right] P(Q)$$

$$= P(Q) + \frac{1}{\varepsilon_d} \cdot P(Q)$$

$$\therefore MR = P(Q) \left[1 + \frac{1}{\varepsilon_d} \right] \quad \text{or} \quad MR = P(Q) \left[1 - \frac{1}{|\varepsilon_d|} \right]$$

$$\text{At profit-maximizing condition } MR(Q) = MC(Q)$$

$$P(Q) \cdot \left[1 - \frac{1}{|\varepsilon_d|} \right] = MC(Q)$$

$$1 - \frac{1}{|\varepsilon_d|} = \frac{MC(Q)}{P(Q)}$$

$$\vdots$$

$$\frac{MC(Q) - P(Q)}{P(Q)} = \frac{-1}{|\varepsilon_d|} \Rightarrow \frac{P(Q) - MC(Q)}{P(Q)} = 1 - \frac{1}{|\varepsilon_d|}$$

\Rightarrow Lerner's index

$$\frac{P(Q) - MC(Q)}{P(Q)} = 1 - \frac{1}{|\varepsilon_d|}$$

In PC, $|\varepsilon_d| = \infty$ because producers are price taker

$$\therefore \frac{P(Q) - MC(Q)}{P(Q)} = \frac{1}{\infty} = 0 \Rightarrow P = MC(Q)$$

In monopoly, $|\varepsilon_d| < \infty \Rightarrow \frac{1}{|\varepsilon_d|} > 0$

$$\therefore \frac{P(Q) - MC(Q)}{P(Q)} > 0 \Rightarrow P(Q) > MC(Q)$$

firm has market power.

5 Effect of taxes

$\Pi(Q) = TR(Q) - TC(Q)$, what will happen to Q^* if government imposes

1. lump-sum tax: $T = t_0$ fixed amount
2. profit tax : $T = t\Pi$ where $0 < t < 1$
3. specific tax : $T = tQ$ where $0 < t < 1$

✓ lump-sum tax

$$\Pi(Q) = TR(Q) - TC(Q) - t_0$$

$$\frac{d}{dQ}\Pi(Q) = MR - MC = 0 \Rightarrow MR = MC \Rightarrow \text{no impact on } Q^*$$

✓ profit tax

$$\Pi(Q) = TR(Q) - TC(Q) - t\Pi = (1-t)(TR-TC)$$

$$\frac{d}{dQ}\Pi(Q) = (1-t)(MR - MC) = 0 \Rightarrow MR = MC \Rightarrow \text{no impact on } Q^*$$

✓ specific tax

$$\Pi(Q) = TR(Q) - TC(Q) - tQ$$

$$\frac{d}{dQ}\Pi(Q) = MR - MC - t = 0 \Rightarrow MR = MC + t \Rightarrow Q^* \text{ will be lower.}$$

6 Maximization of tax revenue

$$P = a - bQ \Rightarrow TR = aQ - bQ^2$$

$$TC = c_0 + c_1Q + c_2Q^2$$

Suppose government imposes tax \$ t per unit;

$$TC \text{ after tax: } TC_T = c_0 + c_1Q + c_2Q^2 + tQ$$

$$\Pi \text{ after tax: } \Pi_T = aQ - bQ^2 - c_0 - c_1Q - c_2Q^2 - tQ$$

Profit-maximizing condition:

$$\text{FOC: } \frac{d}{dQ} \Pi_T = \Pi(Q) = \underbrace{a - 2bQ}_{\text{MR}} - \underbrace{c_1 + 2c_2Q}_{\text{MC}} - t = 0$$

$$a - c_1 - 2(b + c_2)Q - t = 0$$

$$Q^* = \frac{a - c_1 - t}{2(b + c_2)}$$

$$\text{SOC: } \frac{d^2 \Pi_T}{dQ^2} = -2b - 2c_2 < 0 \quad \text{*SOSC is satisfied.}$$

$$\therefore \text{total tax revenue } t \cdot Q^* = \left[\frac{a - c_1 - t}{2(b + c_2)} \right] t$$

Tax-revenue maximizing condition:

$$\text{Max}_t T(t) = \frac{(a - c_1 - t)t}{2(b + c_2)}$$

$$\text{FONC: } \frac{d}{dt} T(t) = \frac{a - c_1 - 2t}{2(b + c_2)} = 0 \Rightarrow t^* = \frac{a - c_1}{2}$$

$$\text{SOSC: } \frac{d^2}{dt^2} T(t) = \frac{-2}{2(b + c_2)} < 0$$

ex. $P = 40 - 0.5Q$, $TC = 2 - 5Q + 7Q^2$; find t (specific tax) that maximizes T .

$$\Rightarrow \text{Max}_Q \Pi_T = (40 - 0.5Q)Q - (2 - 5Q + 7Q^2) - tQ$$

$$\text{FONC: } \frac{d}{dQ} \Pi_T = 40 - Q + 5 - 14Q - t = 0 \Rightarrow 15Q = 45 - t \Rightarrow Q^* = \frac{45 - t}{15}$$

$$\text{SOSC: } \frac{d^2}{dQ^2} \Pi_T = -15 < 0$$

$$\text{Total tax revenue } T = \left(\frac{45 - t}{15} \right) t = \frac{45t - t^2}{15}$$

$$\text{Max}_t T(t) = \frac{45t - t^2}{15}$$

$$\text{FONC: } \frac{d}{dt} T(t) = \frac{45}{15} - \frac{2t}{15} = 0 \Rightarrow t^* = \frac{45}{2} = 22.5$$

$$\text{SOSC: } \frac{d^2}{dt^2} T(t) = \frac{-2}{15} < 0$$