Floor $\lfloor x \rfloor$ and Ceiling $\lceil x \rceil$

Definition (Floor)

Given any real number x, the floor of x, denoted $\lfloor x \rfloor$, is defined as:

$$\lfloor x \rfloor = n, \quad n \text{ is an integer such that } n \le x < n+1.$$

Symbolically, if x is a real number,

$$\boxed{\lfloor x \rfloor = n \;\Leftrightarrow\; n \in \mathbb{Z},\; n \le x < n+1.}$$

n x n+1

x = 6.7

Definition (Ceiling)

Given any real number x, the ceiling of x, denoted $\lceil x \rceil$, is defined as:

$$\lceil x \rceil = n, \quad n \text{ is an integer such that } n-1 < x \le n.$$

Symbolically, if x is a real number,

$$\boxed{\lceil x \rceil = n \;\Leftrightarrow\; n \in \mathbb{Z},\; n-1 < x \le n.}$$

n-1 x n

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Example: Compute $\lfloor x \rfloor$ and $\lceil x \rceil$ of the following values of x.

1. $x = 37.999$ → $\lfloor x \rfloor = 37$, $\lceil x \rceil = 38$
2. $x = 11$ → $\lfloor x \rfloor = 11$, $\lceil x \rceil = 11$
3. $x = 0.6$ → $\lfloor x \rfloor = 0$, $\lceil x \rceil = 1$
4. $x = -\frac{57}{2}$ → $\lfloor -28.5 \rfloor = -29$, $\lceil -28.5 \rceil = -28$
5. $x = -14.001$ → $\lfloor -14.001 \rfloor = -15$, $\lceil -14.001 \rceil = -14$

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Example: Use the proof by a counterexample to show that the statement

"$\forall x, y \in \mathbb{R}, \lfloor x + y \rfloor = \lfloor x \rfloor + \lfloor y \rfloor$"

is **false**.

Answer:

$x = 0 \to \lfloor 0 \rfloor = 0$

$y = -1.5 \quad \lfloor -1.5 \rfloor = -2$

$\lfloor x + y \rfloor = \lfloor -1.5 \rfloor$

A counterexample: $x = 0.5$, $y = 0.5 \to \lfloor 0.5 \rfloor = 0$

$x + y = 1 \to \lfloor 1 \rfloor = 1$

$\therefore \lfloor x + y \rfloor = 1 \neq \underbrace{\lfloor x \rfloor}_{=0} + \underbrace{\lfloor y \rfloor}_{=0} = 0$

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Example: Use the method of proof by cases to prove the following statement.

Let n be an integer. Then

$$\left\lfloor \frac{n}{2} \right\rfloor = \begin{cases} \frac{n}{2}, & \text{if } n \text{ is even} \quad \text{— ①} \\ \frac{n-1}{2} & \text{if } n \text{ is odd} \quad \text{— ②} \end{cases}.$$

Proof by cases:

Case ① : n is even

By definition $\Rightarrow$ $n = 2a$, $a \in \mathbb{Z}$ $\Rightarrow a = \frac{n}{2}$

$\left\lfloor \frac{n}{2} \right\rfloor = \left\lfloor \frac{2a}{2} \right\rfloor = \lfloor a \rfloor = a = \boxed{\frac{n}{2}}$

Case ② n is odd

By definition. $\Rightarrow$ $n = 2b + 1$, $b \in \mathbb{Z} \Rightarrow$

$2b = n - 1$

$b = \frac{n-1}{2}$

$\left\lfloor \frac{n}{2} \right\rfloor = \left\lfloor \frac{2b+1}{2} \right\rfloor = \left\lfloor b + \frac{1}{2} \right\rfloor = b = \frac{n-1}{2}$

$\therefore \left\lfloor \frac{n}{2} \right\rfloor = \frac{n-1}{2}$.

$\therefore$ This formula is true by cases ① and ②.