

Floor $\lfloor x \rfloor$ and Ceiling $\lceil x \rceil$

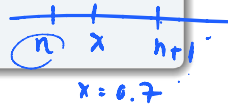
Definition (Floor)

Given any real number x , the floor of x , denoted $\lfloor x \rfloor$, is defined as:

$$\lfloor x \rfloor = n, \quad n \text{ is an integer such that } n \leq x < n+1.$$

Symbolically, if x is a real number,

$$\lfloor x \rfloor = n \Leftrightarrow n \in \mathbb{Z}, n \leq x < n+1.$$



Definition (Ceiling)

Given any real number x , the ceiling of x , denoted $\lceil x \rceil$, is defined as:

$$\lceil x \rceil = n, \quad n \text{ is an integer such that } n-1 < x \leq n.$$

Symbolically, if x is a real number,

$$\lceil x \rceil = n \Leftrightarrow n \in \mathbb{Z}, n-1 < x \leq n.$$



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Example: Compute $\lfloor x \rfloor$ and $\lceil x \rceil$ of the following values of x .

- 1 $x = 37.999 \rightarrow \lfloor x \rfloor = 37, \lceil x \rceil = 38$
- 2 $x = 11 \rightarrow \lfloor x \rfloor = 11, \lceil x \rceil = 11$
- 3 $x = 0.6 \rightarrow \lfloor x \rfloor = 0, \lceil x \rceil = 1$
- 4 $x = -\frac{57}{2} \rightarrow \lfloor x \rfloor = -29, \lceil x \rceil = -28$
- 5 $x = -14.001 \rightarrow \lfloor x \rfloor = -15, \lceil x \rceil = -14$

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Example: Use the proof by a counterexample to show that the statement

$$\forall x, y \in \mathbb{R}, \lfloor x + y \rfloor = \lfloor x \rfloor + \lfloor y \rfloor$$

is false.

Answer:

A counterexample:

$$\begin{aligned} x &= 0.5 \rightarrow \lfloor 0.5 \rfloor = 0 \\ y &= 0.5 \rightarrow \lfloor 0.5 \rfloor = 0 \end{aligned}$$

$$x + y = 1 \rightarrow \lfloor 1 \rfloor = 1$$

$$\therefore \lfloor x + y \rfloor = 1 \neq \underbrace{\lfloor x \rfloor}_{=0} + \underbrace{\lfloor y \rfloor}_{=0} = 0$$

$$\begin{aligned} x &= 0 \rightarrow \lfloor 0 \rfloor = 0 \\ y &= -1.5 \rightarrow \lfloor -1.5 \rfloor = -2 \\ \hline \lfloor x + y \rfloor &= \lfloor -1.5 \rfloor = -2 \end{aligned} \quad \times$$

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Example: Use the method of proof by cases to prove the following statement.

Let n be an integer. Then

$$\left\lfloor \frac{n}{2} \right\rfloor = \begin{cases} \frac{n}{2} & \text{if } n \text{ is even} \quad \text{--- ①} \\ \frac{n-1}{2} & \text{if } n \text{ is odd} \quad \text{--- ②} \end{cases}$$

Proof by cases:

Case ① : n is even
By definition $\Rightarrow n = 2a, a \in \mathbb{Z} \Rightarrow a = \frac{n}{2}$
 $\left\lfloor \frac{n}{2} \right\rfloor = \left\lfloor \frac{2a}{2} \right\rfloor = \lfloor a \rfloor = a = \left\lfloor \frac{n}{2} \right\rfloor$

Case ② : n is odd
By definition $\Rightarrow n = 2b + 1, b \in \mathbb{Z} \Rightarrow$
 $\left\lfloor \frac{n}{2} \right\rfloor = \left\lfloor \frac{2b+1}{2} \right\rfloor = \left\lfloor b + \frac{1}{2} \right\rfloor = b = \frac{n-1}{2}$
 $\therefore \left\lfloor \frac{n}{2} \right\rfloor = \frac{n-1}{2}$

$\therefore$ This formula is true by cases ① and ②.

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