

EE 325 Section 1 (Aj.Wanwiphang) Homework Assignment 1

Due date: 31 January 2020 before 11pm

**** Please submit this assignment on Moodle. For those who work on paper, please scan or submit the pictures of your work. ****

1. Find the answers following questions (please also show your calculation)

$$\begin{aligned} \text{a. } \sum_{i=1}^5 (a + bx_i) &= (a + bx_1) + (a + bx_2) + (a + bx_3) + (a + bx_4) + (a + bx_5) \\ &= 5a + bx_1 + bx_2 + bx_3 + bx_4 + bx_5 \\ &= 5a + b(x_1 + x_2 + x_3 + x_4 + x_5) \quad \text{H} \end{aligned}$$

$$\begin{aligned} \text{b. } \sum_{y=0}^5 f(x+y) &= (x+0) + (x+1) + (x+2) + (x+3) + (x+4) + (x+5) \\ &= 6x + 15 \quad \text{H} \end{aligned}$$

$$\begin{aligned} \text{c. } \sum_{i=1}^{10} i^2 &= 1^2 + 2^2 + 3^2 + 4^2 + 5^2 + 6^2 + 7^2 + 8^2 + 9^2 + 10^2 \\ &= 385 \quad \text{H} \end{aligned}$$

$$\begin{aligned} \text{d. } \sum_{x=1}^2 \sum_{y=2}^3 (2x+y) &= (2(1)+2) + (2(2)+3) \\ &= 11 \quad \text{H} \end{aligned}$$

2. Given X is discrete random variable. The probability distribution function (PDF) of this variable is shown in the table

X	-2	-1	0	1	2	3	4
$f(x)$	0.5b	b	2.25b	2b	1.5b	0.5b	0.25b

** when b is constant number

- a. Find the value of b

$$\begin{aligned} 0.5b + b + 2.25b + 2b + 1.5b + 0.5b + 0.25b &= 1 \\ 8b &= 1 \\ b &= 1/8 = 0.125 \quad \text{H} \end{aligned}$$

- b. Find the answer for $P(X \leq 2)$

$$\begin{aligned} P(X \leq 2) &= 1 - P(X > 2) = 1 - 0.5(0.125) - 0.25(0.125) \\ &= 1 - P(X=3) - P(X=4) = 0.90625 \end{aligned}$$

- c. Find the answer for $P(-2 \leq X \leq 3)$

$$\begin{aligned} P(-2 \leq X \leq 3) &= 1 - P(X > 3) \\ &= 1 - P(X=4) \\ &= 1 - 0.25(0.125) = 0.96875 \end{aligned}$$

- d. Find the answer for $P(X \geq 1)$

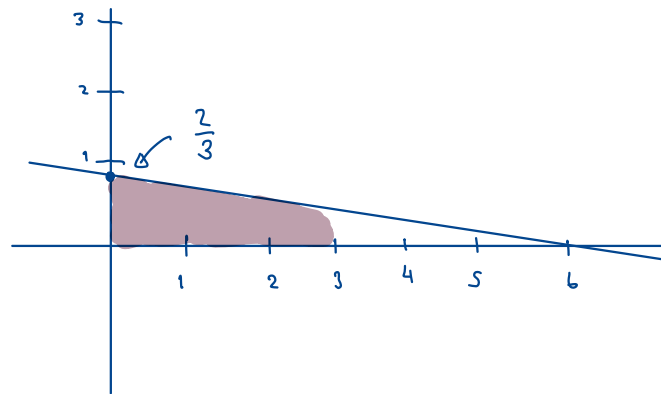
$$\begin{aligned} P(X \geq 1) &= 1 - P(X < 1) \\ &= 1 - P(X=0) - P(X=-1) - P(X=-2) \\ &= 1 - (2.25)(0.125) - (0.125) - 0.5(0.125) = 0.53125 \end{aligned}$$

3. Given X is continuous random variable. The probability distribution function (PDF) of this variable is

$$f(x) = -\frac{1}{9}x + \frac{6}{9}, 0 \leq x \leq 3$$

$(X, Y) = (6, \frac{2}{3})$

- a. Plot graph for $f(x)$



- b. Find the answer for $P(1 \leq X \leq 3)$

$$\begin{aligned} P(1 \leq X \leq 3) &= \int_1^3 f(x) dx \\ &= \left. -\frac{1}{18}x^2 + \frac{6}{9}x \right|_1^3 \\ &= \left(-\frac{(3)^2}{18} + \frac{6}{9}(3) \right) - \left(-\frac{(1)^2}{18} + \frac{6(1)}{9} \right) = -\frac{9}{18} + \frac{18}{9} + \frac{1}{18} - \frac{6}{9} \\ &= -\frac{8}{18} + \frac{12}{9} = \frac{16}{18} \quad \# \end{aligned}$$

- c. Find the answer for $P(X \geq 2)$

$$\begin{aligned} P(X \geq 2) &= \int_2^3 f(x) dx \\ &= \left. -\frac{1}{18}x^2 + \frac{6}{9}x \right|_2^3 \\ &= \left[-\frac{(3)^2}{18} + \frac{6(3)}{9} \right] - \left[-\frac{(2)^2}{18} + \frac{6(2)}{9} \right] = -\frac{9}{18} + \frac{18}{9} + \frac{4}{18} - \frac{12}{9} = \frac{7}{18} \quad \# \end{aligned}$$

- d. Find the expected value of X

$$\begin{aligned} E(X) &= \int_{-\infty}^{\infty} x f(x) dx \\ &= \int_0^3 x \left(-\frac{1}{9}x + \frac{6}{9} \right) dx \\ &= \int_0^3 -\frac{1}{9}x^2 + \frac{6}{9}x dx \\ &= \left. -\frac{x^3}{27} + \frac{6x^2}{18} \right|_0^3 \\ &= -\frac{(3)^3}{27} + \frac{6(3)^2}{18} \\ &= -\frac{27}{27} + \frac{6 \cdot 9}{18} \\ &= -1 + 3 \\ &= 2 \quad \# \end{aligned}$$

4. Let random variable X be the outcome of throwing one dice and random variable Y be the outcome of tossing one coin. Coin has two sided that has valued 1 and 0.

- a. Construct the joint probability distribution function (PDF) table of X and Y

$y \backslash x$	1	2	3	4	5	6
0	$\frac{1}{12}$	$\frac{1}{12}$	$\frac{1}{12}$	$\frac{1}{12}$	$\frac{1}{12}$	$\frac{1}{12}$
1	$\frac{1}{12}$	$\frac{1}{12}$	$\frac{1}{12}$	$\frac{1}{12}$	$\frac{1}{12}$	$\frac{1}{12}$

- b. Find the marginal probability distribution function (PDF) of X

X	1	2	3	4	5	6
$f(x)$	$\frac{2}{12}$	$\frac{2}{12}$	$\frac{2}{12}$	$\frac{2}{12}$	$\frac{2}{12}$	$\frac{2}{12}$

- c. Find the marginal probability distribution function (PDF) of Y

y	0	1
$f(y)$	$\frac{1}{2}$	$\frac{1}{2}$

- d. Find the conditional probability distribution function (PDF) of X given Y is equal to 1

$$f(x|y=1) = \frac{f(x,y)}{f_y(1)} = \frac{f(x,1)}{\frac{1}{2}}$$

X	1	2	3	4	5	6
$f(x y=1)$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$

- e. Find the expected value of X given Y is equal to 1

$$E(X|Y=1) = \sum x_i P(X=x_i|Y=1) = \frac{\sum x_i P(X=x_i, Y=1)}{P(Y=1)} = \frac{1}{P(Y=1)} \sum x_i P(X=x_i, Y=1)$$

$$= \frac{1}{0.5} \left[(1 \cdot \frac{1}{12}) + (2 \cdot \frac{1}{12}) + (3 \cdot \frac{1}{12}) + \dots + (6 \cdot \frac{1}{12}) \right] = \frac{7}{2}$$

- f. Find the variance of X given Y is equal to 1

$$\begin{aligned} V(X|Y=1) &= \sum (X - E(X|Y=1))^2 \cdot P(X|Y=1) \\ &= \left[\left(1 - \frac{7}{2}\right)^2 \cdot \frac{1}{6} + \left(2 - \frac{7}{2}\right)^2 \cdot \frac{1}{6} + \dots + \left(6 - \frac{7}{2}\right)^2 \cdot \frac{1}{6} \right] \\ &= \left(\frac{25}{4} \cdot \frac{1}{6} \right) + \left(\frac{9}{4} \cdot \frac{1}{6} \right) + \left(\frac{1}{4} \cdot \frac{1}{6} \right) + \left(\frac{1}{4} \cdot \frac{1}{6} \right) + \left(\frac{9}{4} \cdot \frac{1}{6} \right) + \left(\frac{25}{4} \cdot \frac{1}{6} \right) \\ &= \frac{120}{24} = \frac{10}{3} \quad \# \end{aligned}$$

5. If X_1, X_2, X_3 is a random sample from a population with mean μ and variance σ^2 . X_1, X_2, X_3 are not independent

$$\text{Cov}(X_1, X_2) = \text{Cov}(X_1, X_3) = \text{Cov}(X_2, X_3) = \frac{1}{4}\sigma^2$$

$\bar{X}$ is estimator used to estimate mean value. $\bar{X} = \frac{1}{3}(X_1 + X_2 + X_3)$

Find $E(\bar{X})$ and $\text{var}(\bar{X})$

$$E(\bar{X}) = E\left(\frac{1}{N} \sum_{i=1}^3 X_i\right)$$

$$= \frac{1}{N} E(X_1 + X_2 + X_3)$$

$$= \frac{1}{N} [E(X_1) + E(X_2) + E(X_3)]$$

$$= \frac{1}{3} [\mu_x + \mu_x + \mu_x]$$

$$= \frac{1}{3} \cdot 3\mu_x = \mu_x$$

$$\begin{aligned} \text{Var}(\bar{X}) &= \text{Var}\left(\frac{1}{N} \sum_{i=1}^3 X_i\right) \\ &= \frac{1}{N^2} \text{Var}(X_1 + X_2 + X_3) \\ &= \frac{1}{N^2} \left[\text{Var}(X_1) + \text{Var}(X_2) + \text{Var}(X_3) \right. \\ &\quad \left. + 2 \text{Cov}(X_1, X_2) + 2 \text{Cov}(X_1, X_3) \right. \\ &\quad \left. + 2 \text{Cov}(X_2, X_3) \right] \\ &= \frac{1}{3^2} \left[\frac{1}{4} \sigma_x^2 + \frac{1}{4} \sigma_x^2 + \frac{1}{4} \sigma_x^2 + \frac{1}{2} \sigma_x^2 \right] \\ &= \frac{1}{3^2} \cdot \frac{9}{4} \sigma_x^2 = \frac{\sigma_x^2}{4} \end{aligned}$$

6. Given X_1, X_2, X_3, X_4 are independent identically distributed random variables from population with mean μ and variance σ^2 . $\bar{X}$ is estimator used to estimate mean value. $\bar{X} = \frac{1}{4}(X_1 + X_2 + X_3 + X_4)$

a. Find $E(\bar{X})$ and $\text{var}(\bar{X})$ in term of μ and σ

$$E(\bar{X}) = E\left(\frac{1}{N} \sum_{i=1}^4 X_i\right)$$

$$= \frac{1}{N} E(X_1 + X_2 + X_3 + X_4)$$

$$= \frac{1}{N} [E(X_1) + E(X_2) + E(X_3) + E(X_4)]$$

$$= \frac{1}{4} [\mu_x + \mu_x + \mu_x + \mu_x]$$

$$= \frac{1}{4} \cdot 4\mu_x = \mu_x$$

$$\begin{aligned} \text{Var}(\bar{X}) &= \text{Var}\left(\frac{1}{N} \sum_{i=1}^4 X_i\right) \\ &= \frac{1}{N^2} \text{Var}(X_1 + X_2 + X_3 + X_4) \\ &= \frac{1}{N^2} [\text{Var}(X_1) + \text{Var}(X_2) + \text{Var}(X_3) + \text{Var}(X_4)] \\ &= \frac{1}{4^2} [\sigma_x^2 + \sigma_x^2 + \sigma_x^2 + \sigma_x^2] \\ &= \frac{1}{16} \cdot 4\sigma_x^2 = \frac{\sigma_x^2}{4} = 0.25\sigma_x^2 \end{aligned}$$

- b. Given $\tilde{X} = \frac{1}{8}X_1 + \frac{1}{4}X_2 + \frac{1}{8}X_3 + \frac{1}{2}X_4$ is another estimator of μ . Show that $\tilde{X}$ is an unbiased estimator of μ

$$\begin{aligned}\tilde{X} &= \frac{1}{4}(0.5X_1 + X_2 + 0.5X_3 + 2X_4) \\ E(\tilde{X}) &= E\left(\frac{1}{4}\sum_{i=1}^4 X_i\right) \\ &= \frac{1}{4}E(0.5X_1 + X_2 + 0.5X_3 + 2X_4) \\ &= \frac{1}{4}[E(0.5X_1) + E(X_2) + E(0.5X_3) + E(2X_4)] \\ &= \frac{1}{4}[0.5E(X_1) + E(X_2) + 0.5E(X_3) + 2E(X_4)] \\ &= \frac{1}{4}(0.5\mu_X + \mu_X + 0.5\mu_X + 2\mu_X) \\ &= \frac{1}{4} \cdot 4\mu_X = \mu_X\end{aligned}$$

$\tilde{X}$ is unbiased estimator of μ

$$\begin{aligned}\text{Var}(\tilde{X}) &= \text{Var}\left(\frac{1}{4}\sum_{i=1}^4 X_i\right) \\ &= \frac{1}{4^2}\text{Var}(0.5X_1 + X_2 + 0.5X_3 + 2X_4) \\ &= \frac{1}{4^2}[\text{Var}(0.5X_1) + \text{Var}(X_2) + \text{Var}(0.5X_3) + \text{Var}(2X_4)] \\ &= \frac{1}{4^2}[0.25\text{Var}(X_1) + \text{Var}(X_2) + 0.25\text{Var}(X_3) + \text{Var}(X_4)] \\ &= \frac{1}{4^2}[0.25\sigma_X^2 + \sigma_X^2 + 0.25\sigma_X^2 + 4\sigma_X^2] \\ &= \frac{5.5\sigma_X^2}{16} = 0.34\sigma_X^2\end{aligned}$$

- c. Between $\bar{X}$ and $\tilde{X}$, which one is the better estimator for μ ? Why?

$\bar{X}$ is more efficient of μ_X than $\tilde{X}$ due to smaller variance.