Chapter 4 Integration

Section 4.1 Antiderivatives

Definition: A function *F* is called an *antiderivative* of *f* on an interval *I* if $F'(x) = f(x)$ for all *x* in *I*.

For instance, let $f(x) = x^2$. We know that if $F(x) = (1/3)x^3$, then $F'(x) = x^2 = f(x)$ .But the function $G(x) = (1/3)x^3 + 75$ also satisfies $G'(x) = x^2$.Therefore, both *F* and *G* are antiderivative of *f*. Indeed, any function of the form $H(x) = (1/3)x^3 + C$, where C is a constant, is an antiderivative of *f*.

Thus, if F and G are any two antiderivatives of f, then $F' = f(x) = G'(x)$ so $G(x) - F(x) = C$, where C is a constant. We can write this as $G(x) = F(x) + C$, so we have the following result.

Theorem: If *F* is an antiderivative of *f* on an interval *I*, then the most general antiderivative of f on *I* is

$$F(x) + c$$

where *c* is an arbitrary constant.

Table of Antidifferentiation Formulas

Function	Particular antiderivative
$cf(x)$	$cF(x)$
$f(x)+g(x)$	$F(x)+G(x)$
x^n $(n \neq -1)$	$(x^{n+1})/(n+1)$
$\cos x$	$\sin x$
$\sin x$	$-\cos x$
$\sec^2 x$	$\tan x$
$\sec x \tan x$	$\sec x$

Note : To obtain the most general antiderivative from the particular ones, we have to add a constant (or constants).

Example 1: Find all functions *f* such that $f'(x) = 8x^3 + 12x + 3$.

Example 2: Find *f* where $f'(x) = 8x^3 + 12x + 3$, $f(1) = 6$.

Section 4.2 The Definite Integral

Definition of a Definite Integral :

If f is a continuous function defined for a ≤ x ≤ b, we divide the interval [a,b] into n subintervals of equal width $\Delta x = (b-a)/n$ .We let $x_0(=a)$, x_1, x_2,…,$x_n(=b)$ be the endpoints of these subintervals and we let $x_1^*, x_2^*, ..., x_n^*$ be any sample points in these subintervals, so x_i^* lies in the *i*th subinterval $[x_{i-1}, x_i]$. Then the definite integral of *f* from *a* to *b* is

$$\int_a^b f(x)dx = \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i^*)\Delta x$$

Because we have assumed that *f* is continuous, it can be proved that the limit in the definition above always exists and gives the same value no matter how we choose the sample points x_i^*. If we take the sample points to be right endpoints, then $x_i^* = x_i$ and the definition of an integral becomes

$$\int_a^b f(x)dx = \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i)\Delta x$$

If we choose the sample points to be left endpoints, then $x_i^* = x_{i-1}$ and the definition becomes

$$\int_a^b f(x)dx = \lim_{n\to\infty}\sum_{i=1}^{n} f(x_{i-1})\Delta x$$

Alternatively, we could choose x_i^* to be the midpoint of the subinterval or any other number between x_{i-1} and x_i.

Note 1: The symbol $\int$ was introduced by Leibniz and is called an integral sign. It is an elongated *S* and was chosen because an integral is a limit of sums. In the notation $\int_a^b f(x)dx, \int f(x)dx$ is called the **integrand** and *a* and *b* are called the **limits of integration**; *a* is the **lower limit** and *b* is the **upper limit**. The symbol *dx* has no official meaning by itself; $\int_a^b f(x)dx$ is all one symbol. The procedure of calculating an integral is called integration.

Note 2: The definite integral $\int_a^b f(x)dx$ is a number; it does not depend on *x*. In fact, we could use any letter in place of *x* without changing the value of the integral.

$$\int_a^b f(x)dx = \int_a^b f(t)dt = \int_a^b f(r)dr$$

Note 3: The sum

$$\sum_{i=1}^{n} f(x_i^*)\Delta x$$

that occurs in the definition above is called a **Riemann sum** after the German mathematician Bernhard Riemann (1826-1866). We know that if *f* happens to be positive, the Riemann sum can be interpreted as a sum of areas of approximating rectangles (see figure 4.1(a)). By comparing the definition above with the definition of area, we see that the definite integral $\int_a^b f(x)dx$ can be interpreted as the area under the curve *y* = *f*(*x*) from *a* to *b*. (See figure 4.1(b)).

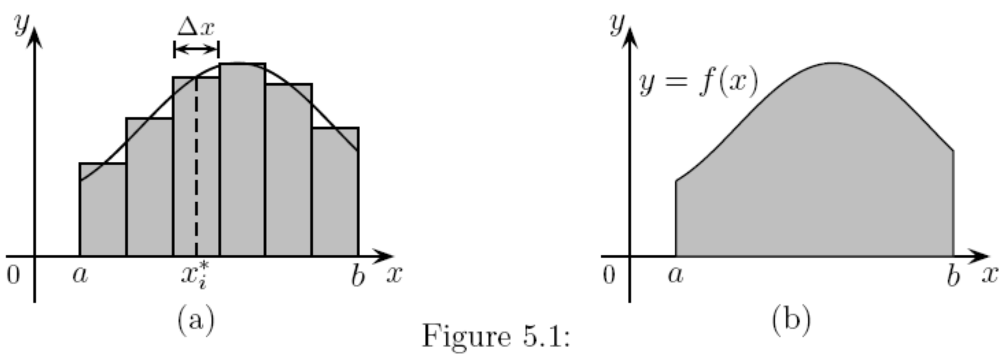
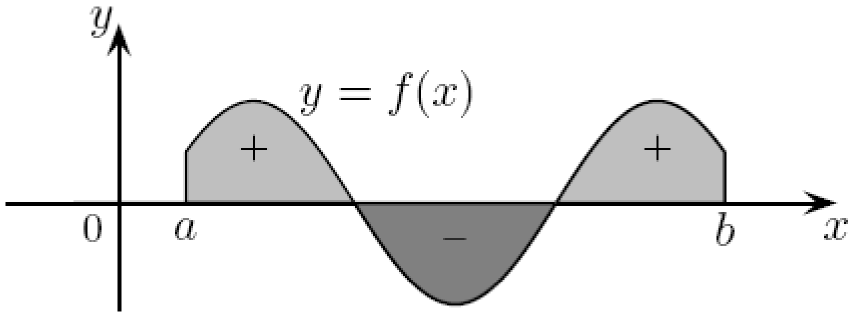


Figure 5.1:

If *f* takes on both positive and negative values then the Riemann sum is the sum of the areas of the rectangles that lie above the *x*-axis and the *negatives* of the areas of the rectangles that lie below the *x*-axis. When we take the limit of such Riemann sums, we get the situation illustrated in figure below.



A definite integral can be interpreted as a net area, that is, a difference of areas:

$$\int_a^b f(x)dx = A_1 - A_2$$

where A_1 is the area of the region above the *x*-axis and below the graph of *f*, and A_2 is the area of the region below the *x*-axis and above the graph of *f*.

Note 4: Although we have defined $\int_a^b f(x)dx$ by dividing [*a*,*b*] into subintervals of equal width, there are situations in which it is advantageous to work with subintervals of unequal width.

Properties of the Definite Integral

$$(1)\int_b^a f(x)dx = -\int_a^b f(x)dx$$

$$(2)\int_a^a f(x)dx = 0$$

Properties of the Integral

$$(1)\int_a^b c\,dx = c(b-a) \text{ where } c \text{ is any constant}$$

$$(2)\int_a^b \left[f(x)+g(x)\right]dx = \int_a^b f(x)dx + \int_a^b g(x)dx$$

$$(3)\int_a^b \left[f(x)-g(x)\right]dx = \int_a^b f(x)dx - \int_a^b g(x)dx$$

$$(4)\int_a^b cf(x)dx = c\int_a^b f(x)dx \quad \text{where } c \text{ is any constant}$$

$$(5)\int_a^c f(x)dx + \int_c^b f(x)dx = \int_a^b f(x)dx$$

Comparison Properties of the Integral

(6)If $f(x) \ge 0$ for $a \le x \le b$, then $\int_a^b f(x)dx \ge 0$

(7)If $f(x) \ge g(x)$ for $a \le x \le b$,

then $\int_a^b f(x)dx \ge \int_a^b g(x)dx$

(8)If $m \le f(x) \le M$ for $a \le x \le b$,

then $m(b-a) \le \int_a^b f(x)dx \le M(b-a)$

Section 4.3 The Fundamental Theorem of Calculus

<u>The Fundamental Theorem of Calculus, Part I (FTC I)</u>

If *f* is continuous on [a,b], then the function *g* defined by

$$g(x) = \int_a^x f(t)dt \quad a \le x \le b$$

is continuous on [a,b] and differentiable on (a,b), and $g'(x) = f(x)$.

<u>The Fundamental Theorem of Calculus, Part II (FTC II)</u>

If *f* is continuous on [a,b], then

$$\int_a^b f(x)dx = F(b) - F(a)$$

where F is any antiderivative of *f*, that is, $F'(x) = f(x)$

Basically,

Part I says “derivatives” and “integrals” are opposite (or inverse operations).

Part II gives us (finally) a method to evaluate definite integrals without using the limit.

<u>Example 1</u>: Find the derivative of the function.

$1.\ g(x)=\int_0^x \cos t\,dt \qquad 2.\ g(x)=\int_0^x \sin^2 t\,dt$

$3.\ g(t)=\int_0^t x^3\,dx \qquad 4.\ g(x)=\int_0^{x^4} \sec t\,dt$

$5.\ y=\int_1^{\cos x} (1+\sin t)\,dt \qquad 6.\ y=\int_{1/x^2}^{0} \sin^3 t\,dt$

Example 2: Evaluate the integral.

1. $\int_{1}^{2}\left(x^{3}-x^{2}\right)dx$
2. $\int_{-\pi/2}^{\pi/2}\cos x\,dx$
3. $\int_{0}^{1}\left(2x-1\right)^{2}dx$
4. $\int_{-2}^{5}6\,dx$
5. $\int_{-2}^{3}\left(x^{-4}\right)dx$
6. $\int_{0}^{\pi/6}\csc\theta\cot\theta\,d\theta$
7. $\int_{-\pi}^{\pi}f(x)dx$ where $f(x)=\begin{cases} x & \text{if } -\pi\le x\le 0 \\ \sin x & \text{if } 0<x\le\pi \end{cases}$
8. $\int_{0}^{\pi/3}\frac{1-\sin^{2}x}{\cos^{4}x}dx$

Section 4.4 Indefinite Integrals

Recall :

FTC I : $\frac{d}{dx}\int_a^x f(t)dt = f(x)$

FTC II : $\int_a^b f(x)dx = F(b) - F(a)$ where $F'(x) = f(x)$

The notation $\int f(x)dx$ is traditionally used for an antiderivative of f and is called an indefinite integral. Thus $\int f(x)dx = F(x)$ means $F'(x) = f(x)$

Note: A definite integral $\int_a^b f(x)dx$ is a number, whereas an indefinite integral $\int f(x)dx$ is a function (or family of functions).

Table of Indefinite Integral

(1) $\int cf(x)dx = c\int f(x)dx$

(2) $\int [f(x)+g(x)]dx = \int f(x)dx + \int g(x)dx$

(3) $\int kdx = kx + C$

(4) $\int x^n dx = \frac{x^{n+1}}{n+1} + C \quad (n \neq -1)$

(5) $\int \sin x dx = -\cos x + C$

(6) $\int \cos x dx = \sin x + C$

(7) $\int \sec^2 x dx = \tan x + C$

(8) $\int \csc^2 x dx = -\cot x + C$

(9) $\int \sec x \tan x dx = \sec x + C$

(10) $\int \csc x \cot x dx = -\csc x + C$

(11) $\int \frac{1}{\sqrt{1-x^2}} dx = \sin^{-1} x + C$

(12) $\int \frac{1}{1+x^2} dx = \tan^{-1} x + C$

(13) $\int \frac{1}{x\sqrt{x^2-1}} dx = \sec^{-1} x + C$

Example : Evaluate

1. $\int \sqrt[3]{x}\,dx$ 2. $\int \frac{\sin 2x}{\sin x}\,dx$ 3. $\int_1^2 \frac{y+5y^7}{y^3}\,dy$ 4. $\int_0^{\pi/3} \frac{\sin x + \sin x \tan^2 x}{\sec^2 x}\,dx$ 5. $\int_0^{3\pi/2} |\sin x|\,dx$

Section 4.5 Integration by Substitution

Section 4.6 Evaluating Definite Integrals by Substitution

Consider the function $f(x) = 5(x^2 + 3x)^4 (2x + 3)$. To find the antiderivative F(x) we want F′(x) = $f(x)$, it looks as if the Chain Rule was applied to F(x) to find $f(x)$. Try to reverse it to get $F(x) = (x^2 + 3x)^5 + C$. Then $F'(x) = 5(x^2 + 3x)^4 (2x + 3) = f(x)$.

What about $f(x) = 3xsin(x^2),\ F(x) = ?$

An initial guess might be $F(x) = -cos(x^2) + C,\ F'(x) = 2xsin(x^2) \neq f(x)$!!!

What is missing is a factor of 3/2 , however since constant multiplies are not affected by integration, we can simply adjust after the fact.

$$\begin{aligned} F(x) &= -(3/2)cos(x^2) + C \\ F'(x) &= (-3/2)(2x)(-sin(x^2)) = 3xsin(x^2) = f(x) \end{aligned}$$

This technique enables us to integrate function via a “reverse chain rule” by involving a new variable which the integral is re-written.

Example: $\int x^2 \sin(x^3)dx = ?$

- First check that it does look like a reverse chain rule. ie. Look if the derivative of the “inside” appears on the “outside” (apart from a multiplicative constant).
- Then substitute “u” as the entire inside function.

  Here $u = x^3$, we have to have everything as one variable, so we find du /dx = $3x^2$ → du = $3x^2$dx

Now substitution

$$\int x^2 \sin(x^3)dx = \int \sin(x^3)x^2 dx = \int \sin(u)\frac{1}{3}du = \frac{1}{3}\int \sin(u)du = -\frac{1}{3}\cos u + C = -\frac{1}{3}\cos x^3 + C$$

The Substitution Rule:

If $u = g(x)$ is a differentiable function whose range is an interval I and f is continuous on I, then

$$\int f(g(x))g'(x)dx = \int f(u)du$$

Example 1: Find

1. $\int \frac{\sin(t^{-2}+1)}{t^3}dt$
2. $\int y^3\sqrt{2y^4-1}\,dy$

The Substitution Rule for Definite Integral:

If $g'(x)$ is continuous on [a,b] and f is continuous on the range $u = g(x)$,then

$$\int_a^b f(g(x))g'(x)dx = \int_{g(a)}^{g(b)} f(u)du$$

Example 2: Find $\int_0^{\pi/2} \sin x \cos x dx$

Example 3: Find

$1.\int_2^4 \frac{dx}{(2x-3)^2}$ $2.\int_0^4 \frac{dx}{(2x-3)^2}$ $3.\int_0^7 \sqrt{4+3x}\,dx$ $4.\int_{1/6}^{1/2} \csc(\pi t)\cot(\pi t)dt$

$5.\int_0^{\pi/2} \cos x \sin(\sin x)dx$ $6.\int \sin^3 x \cos x dx$ $7.\int \frac{\cos\sqrt{x}}{\sqrt{x}}dx$ $8.\int x^3(x^2+1)^{99}\,dx$

$9.\int \frac{1}{\sqrt{1-4x^2}}\,dx$ $10.\int \frac{1}{x^2+a^2}\,dx$ $11.\int \frac{x}{x^4+9}\,dx$