Assignment 4

DUE DATE: Tuesday 9^{nd}, March 2021.

I pledge to the Honor Code and to obey all rules for taking and performing homework assignments as specified by the course instructor.

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Question 1 (50 points) Your score......................

Given the daily log returns : (R_t) can be explained by the AR(2) model as following:

$$(1-1.5B+0.9B^2)R_t = 0.25+\varepsilon_t$$

where ε_t is distributed as the Gaussian White Noise with mean (μ) = 0 and variance (σ^2) =0.25
B lag-operator

Question 1.1 (10 points) Your score......................

From the above AR(2) model, Is the model weakly stationary? Write down the reverse characteristic equation and find out the conditions to support your answer.

From above AR(2) model, $(1-1.5B+0.9B^2)R_t = 0.25+\varepsilon_t$

Reversed characteristic equation : $x^2 - 1.5x + 0.9 = 0.25 + \varepsilon_t$

$$\lambda_i = \frac{-(-1.5) \pm \sqrt{(-1.5)^2 - 4(1)(0.9)}}{2}$$

$$\lambda_i = \frac{1.5}{2} \pm \frac{\sqrt{-1.35}}{2}$$

$$= 0.75 \pm \frac{\sqrt{1.35}}{2} i$$

$$R = \sqrt{(0.75)^2 + \left(\frac{\sqrt{1.35}}{2}\right)^2} = 0.9486 < 1$$

Therefore, R_t is weakly stationarity.

Question 1.2 (10 points)

Your score......................

Calculate the unconditional mean: $E(R_t)$ of R_t and the conditional mean: $E(R_t|F_{t-1})$

Conditional mean

$R_t = 0.25 + 1.5R_{t-1} - 0.9R_{t-2} + \varepsilon_t$

$E(R_t|F_{t-1}) = E(0.25|F_{t-1}) + E[1.5R_{t-1}|F_{t-1}] + E[-0.9R_{t-2}|F_{t-1}] + \cancel{E[\varepsilon_t|F_{t-1}]}^{0}$

$E(R_t|F_{t-1}) = 0.25 + 1.5R_{t-1} - 0.9R_{t-2}$

Unconditional mean

$E(R_t) = 0.25 + 1.5E(R_{t-1}) - 0.9E(R_{t-2}) + \cancel{E(\varepsilon_t)}^{0}$

since R_t is weakly stationarity, $E(R_t) = E(R_{t-1}) = E(R_{t-2})$

$E(R_t) - 1.5E(R_t) + 0.9E(R_t) = 0.25$

$(1 - 1.5 + 0.9)\ E(R_t) = 0.25$

$E(R_t) = \dfrac{0.25}{0.4} = 0.625$

Question 1.3 (10 points) Your score......................

Find out the unconditional variance: $Var(R_t)$ of R_t and conditional variance $Var(R_t|F_{t-1})$ of R_t

Conditional variance

$$R_t = 0.25 + 1.5R_{t-1} - 0.9R_{t-2} + \varepsilon_t$$

$$\begin{aligned} Var[R_t | F_{t-1}] &= \overset{0}{\cancel{Var(1.5R_{t-1} | F_{t-1})}} + \overset{0}{\cancel{Var(-0.9R_{t-2} | F_{t-1})}} + Var(\varepsilon_t | F_{t-1}) \\ &\quad + 2\,Cov(1.5R_{t-1}, -0.9R_{t-2} | F_{t-1})^{=0} + 2\,Cov(1.5R_{t-1}, \varepsilon_t | F_{t-1})^{=0} \\ &\quad + 2\,Cov(-0.9R_{t-2}, \varepsilon_t | F_{t-1})^{=0} \\ &= Var(\varepsilon_t | F_{t-1}) = 0.25 \end{aligned}$$

unconditional variance

$$Var(R_t) = \frac{\sigma^2 + 2\phi_1\phi_2\gamma_1}{1 - \phi_1^2 - \phi_2^2} = \frac{(0.25)^2 + 2(1.5)(-0.9)(0.7895)}{1 - (1.5)^2 - (-0.9)^2} \approx 1.0044$$

$$\gamma_1 = \phi_1\gamma_0 + \phi_2\gamma_{-1}$$

$$\gamma_1 = \phi_1\gamma_0 + \phi_2\gamma_1$$

$$\gamma_1 = \frac{\phi_1}{1 - \phi_2} = \frac{1.5}{1 + 0.9} = 0.7895$$

Question 1.4 (10 points) Your score......................

Calculate the autocorrelation: ρ_l for l=1 and 2 of R_t. Also, write down the autocorrelation: ρ_l when l $\geq$ 2.

$(r_t - \mu) = \phi_1 (r_{t-1} - \mu) + \phi_2 (r_{t-2} - \mu) + \varepsilon_t$

Multiply $(r_{t-j} - \mu)$ on both sides and take $E[\cdot]$

$\gamma_j = E[(r_t - \mu)(r_{t-j} - \mu)] = \phi_1 E[(r_{t-1} - \mu)(r_{t-j} - \mu)] + \phi_2 E[(r_{t-2} - \mu)(r_{t-j} - \mu)] + \overset{0}{\cancel{E[\varepsilon_t (r_{t-j} - \mu)]}}$

$\gamma_j = \phi_1 \gamma_{j-1} + \phi_2 \gamma_{j-2}$

$\hookrightarrow \frac{\gamma_j}{\gamma_0} = \phi_1 \frac{\gamma_{j-1}}{\gamma_0} + \phi_2 \frac{\gamma_{j-2}}{\gamma_0}$

$\rho_j = \phi_1 \rho_{j-1} + \phi_2 \rho_{j-2}$ when $j \geq 2$

$\rho_1 = \frac{\gamma_1}{\gamma_0} = \frac{0.7895}{1.0044} = 0.786$

$\rho_2 = \frac{\gamma_2}{\gamma_0} = \frac{\phi_1 \gamma_1 + \phi_2 \gamma_0}{\gamma_0} = 1.5(0.786) + (-0.9) = 0.279$

Question 1.5 (10 points) | Your score......................

Given $R_{1000} = 0.01$ $R_{999} = 0.02$ $R_{998} = 0.03$ $\varepsilon_{1000} = -0.01$ $\varepsilon_{999} = -0.02$ $\varepsilon_{998} = -0.03$ Obtain 1-step, 2-setp 95 % interval forecasts for R_t at the forecast origin t = 1000. Also the ∞-step 95 % interval forecasts for R_t. Draw these intervals.

$R_t = 0.25 + 1.5R_{t-1} - 0.9R_{t-2} + \varepsilon_t$

1- step ahead

$E(R_{h+1} | F_h) = \hat{R}_h(1) = E[0.25 + 1.5R_h - 0.9R_{h-1} + \varepsilon_{h+1} | \cdot]$

$\hat{R}_h(1) = 0.25 + 1.5R_h - 0.9R_{h-1}$

$\hat{R}_h(1) = 0.25 + 1.5(0.01) - 0.9(0.02)$

$\hat{R}_h(1) = 0.247$

Forecasting error : $R_{h+1} - \hat{R}_h(1) = e_h(1) = \varepsilon_{h+1}$

Variance of forecasting error : $Var(e_h(1)|\cdot) = \sigma_\varepsilon^2 = 0.25$

Interval forecasting : $0.247 \pm (1.96)\sqrt{0.25} = (-0.733, 1.227)$

2- step ahead

$\hat{R}_h(2) = \phi_0 + \phi_1 R_{h+1} + \phi_2 R_h$

$= 0.25 + 1.5(0.247) - 0.9(0.01)$

$\hat{R}_h(2) = 0.6115$

Forecasting error : $e_h(2) = R_{h+2} - \hat{R}_h(2) = \phi_1 [R_{h+1} - \hat{R}_h(1)] + \varepsilon_{h+2} = \phi_1 \varepsilon_{h+1} + \varepsilon_{h+2}$

Variance of forecasting error : $Var(e_h(2)|\cdot) = \phi_1^2 Var(\varepsilon_{h+1}|\cdot) + Var(\varepsilon_{h+2}|\cdot) + 2\,cov(\phi_1 \varepsilon_{h+1}, \varepsilon_{h+2}|\cdot)$

$Var(e_h(2)|\cdot) = (1.5)^2 \sigma_\varepsilon^2 + \sigma_\varepsilon^2 = (1.5)^2(0.25) + 0.25 = 2.25$

Interval forecasting : $0.6115 \pm 1.96\sqrt{2.25} = (-2.3285, 3.5515)$

ℓ- step ahead

$\hat{R}_h(\ell) = \phi_0 + \phi_1 R_{h+\ell-1} + \phi_2 R_{h+\ell-2}$

$\lim_{\ell \to \infty} \hat{R}_h(\ell) = mean = E(R_t)$, $\lim_{\ell \to \infty} var(e_h(\ell)) = Var(R_t)$

Interval forecasting: $E(R_t) \pm 1.96\sqrt{Var(R_t)} = 0.625 \pm 1.96\sqrt{1.0044} = (-1.3393, 2.5893)$