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1. Two individuals agree at date 0 to a forward contract that matures at date 2. The contract is written on an underlying asset that pays a dividend at date 1 equal to D_1. Let f_2 be the date 2 random payoff (profit) to the individual who is the long party in the forward contract. Also let m_{0i} be the stochastic discount factor over the period from dates 0 to i where $i = 1, 2$, and let $E_0[\cdot]$ be the expectations operator at date 0. What is the value of $E_0[m_{02} f_2]$? Explain your answer.

1. S_i: Price of the underlying asset at date i

D_0: PV of Div between dates 0 and 1

$$S_0 = E_0[m_{01} D_1] + E_0[m_{02} S_2] = D_0 + E_0[m_{02} S_2] \quad (1)$$

Forward price

payoff $f_2 = S_2 - F_{02}$

Stochastic discount factor approach, $E_0[m_{02} f_2] = E_0[m_{02}(S_2 - F_{02})] = E[m_{02} S_2] - E[m_{02} F_{02}]$

(2) $E[m_{02} F_{02}] = E[m_{02}] F_{02} = R_f^{-2} F_{02}$

$S_0 = E[m_{02} F_{02}] = E[m_{02}] F_{02} = R_f^{-2} F_{02}$

$$E_0[m_{02} f_2] = E[m_{02} S_2] - E[m_{02} F_{02}]$$
$$= S_0 - D_0 - R_f^{-2} F_{02}$$

No arbitrage implies that $F_{02} = R_F^2 (S_0 - D_0)$ and $E[m_{02} f_2] = 0$

2. Assume that there is an economy populated by infinitely-lived representative individuals who maximize the lifetime utility function

$$E_0 \left[\sum_{t=0}^{\infty} -\delta^t e^{-ac_t} \right]$$

where c_t is consumption at date t and $a > 0$, $0 < \delta < 1$. The economy is a Lucas (1978) endowment economy having multiple risky assets paying date t dividends that total d_t per capita. Write down an expression for the equilibrium per capita price of the market portfolio in terms of the assets' future dividends.

2. Note, $c_t^* = d_t^*$ by assumption and $U_c(c_t, t) = + a \delta^t e^{-ac_t}$

$$p_0 = E_0 \left[\sum_{t=1}^{\infty} \frac{U_c(c_t)^*}{U_c(c_0^*)} d_t \right]$$

$$= E_0 \left[\sum_{t=1}^{\infty} \frac{a \delta^t e^{-ad_t}}{a e^{-ad_0}} d_t \right]$$

$$= E_0 \left[\sum_{t=1}^{\infty} \delta^t e^{-a(d_t - d_0)} d_t \right]$$

3. For the Lucas model with labor income, show that assumptions (6.25) and (6.26) lead to the pricing relationship (6.27) and (6.28).

3.
$$P_t = E_t\left[\sum_{t=1}^{\infty} \delta^j \left(\frac{C^*_{t+j}}{C^*_t}\right)^{\gamma-1} d_{t+j}\right]$$

$$\frac{P_t}{d_t} = E_t\left[\sum_{j=1}^{\infty} \delta \left(\frac{C^*_{t+j}}{C^*_t}\right)^{\gamma-1}\left(\frac{d_{t+j}}{d_t}\right)\right]$$

$$= E_t\left[\sum_{j=1}^{\infty} \delta^j e^{(\gamma-1)\ln\left(\frac{C_{t+j}}{C_t}\right) + \ln\left(\frac{d_{t+j}}{d_t}\right)}\right]$$

$$\ln\left(\frac{C_{t+j}}{C_t}\right) = j\mu_c + \sigma_c \sum_{i=1}^{j} \eta_{t+j}$$

$$\ln\left(\frac{D_{t+j}}{D_t}\right) = j\cdot\mu_d + \sigma \sum_{i=1}^{j} \varepsilon_{t+j}$$

$\therefore$
$$\frac{P_t}{d_t} = E_t\left[\sum_{j=1}^{\infty} \delta^j e^{(\gamma-1)\left(j\mu_c + \sigma_c \sum_{i=1}^{j} \eta_{t+j}\right) + j\mu_d + \sigma_d \sum_{i=1}^{j} \varepsilon_{t+j}}\right]$$

$$= E_t\left[\sum_{j=1}^{\infty} \delta^j e^{j((\gamma-1)\mu_c + \mu_c) + \sum_{i=1}^{j}\left[(\gamma-1)\sigma_c \eta_{t+j} + \sigma_d \varepsilon_{t+j}\right]}\right]$$

$$= \sum_{j=1}^{\infty} \delta^j e^{j[(\gamma-1)\mu_c + \mu_d]}\, e^{\frac{j}{2}[(1-\gamma)^2\sigma_c^2 + \sigma_c^2 - 2(1-\gamma)\sigma_c\sigma_d\rho]}$$

$$= \sum_{j=1}^{\infty} e^{j[\ln\delta - (1-\gamma)\mu_c + \mu_d + \frac{1}{2}((1-\gamma)^2\sigma_c^2 + \sigma_d^2) - (1-\gamma)\sigma_c\sigma_d\rho]}$$

$$= \frac{1}{1 - \delta e^{-(1-\gamma)\mu_c + \mu_d + \frac{1}{2}((1-\gamma)^2\sigma_c^2 + \sigma_d^2) - (1-\gamma)\sigma_c\sigma_d\rho}} - 1$$

So
$$P_t = d_t \frac{\delta e^{\alpha}}{1 - \delta e^{\alpha}}$$

Where $\alpha = \mu_d - (1-\gamma)\mu_c + \frac{1}{2}[(1-\gamma)^2\sigma_c^2 + \sigma_d^2] - (1-\gamma)\rho\sigma_c\sigma_d$

4. Consider a special case of the model of rational speculative bubbles discussed in this chapter. Assume that infinitely-lived investors are risk-neutral and that there is an asset paying a constant, one-period risk-free return of $R_f = \delta^{-1} > 1$. There is also an infinitely-lived risky asset with price p_t at date t. The risky asset is assumed to pay a dividend of d_t which is declared at date t and paid at the end of the period, date $t+1$. Consider the price $p_t = f_t + b_t$ where

$$f_t = \sum_{i=0}^{\infty} \frac{E_t\left[d_{t+i}\right]}{R_f{}^{i+1}} \tag{1}$$

and

$$b_{t+1} = \begin{cases} \frac{R_f}{q_t} b_t + e_{t+1} & \text{with probability } q_t \\ z_{t+1} & \text{with probability } 1 - q_t \end{cases} \tag{2}$$

where $E_t\left[e_{t+1}\right] = E_t\left[z_{t+1}\right] = 0$ and where q_t is a random variable as of date $t-1$ but realized at date t and is uniformly distributed between 0 and 1.

4.a Show whether or not $p_t = f_t + b_t$ subject to the specifications in (1) and (2) is a valid solution for the price of the risky asset.

4.b Suppose that p_t is the price of a barrel of oil. If $p_t \geq p_{solar}$, then solar energy, which is in perfectly elastic supply, becomes an economically efficient perfect substitute for oil. Can a rational speculative bubble exist for the price of oil? Explain why or why not.

4.c Suppose p_t is the price of a bond that matures at date $T < \infty$. In this context, the d_t for $t \leq T$ denotes the bond's coupon and principal payments. Can a rational speculative bubble exist for the price of this bond? Explain why or why not.

4a.)

(2) must satisfy $E_t[b_{t+1}] = \delta^{-1} b_t$ for $P_t = f_t + b_t$ to be a valid solution

$$E_t[b_{t+1}] = \frac{R_f}{q_t} b_t q_t + E_t[e_{t+1}] q_t + E_t[z_{t+1}](1-q_t)$$
$$= R_f b_t = \delta^{-1} b_t.$$

∴ Valid.

4b.) P_t must be lower than P_{solar} because of perfect substitute.

$$\lim_{i \to \infty} \delta^{-i} b_t = \begin{cases} \infty & \text{if } b_t > 0 \\ -\infty & \text{if } b_t < 0 \end{cases}$$

and certainly not $-\infty$

The price of oil cannot fall below 0. Moreover, it cannot go to infinity since it cannot exceed P_{solar} so it must be in a finite value.

∴ The rational bubble doesn't exist for oil.

4c) The price of bond cannot fall below 0 and certainly not negative infinity. Moreover, the price of bond is d_T at T and 0 afterward. Therefore, the price could not go to beyond ∞.

∴ The rational bubble does not exist for the bond.

5. Consider an endowment economy with representative agents who maximize the following objective function:

$$\max_{C_s, \{\omega_{is}\}, \forall s, i} E_t \left[\sum_{s=t}^{T} \delta^s u(C_s) \right]$$

where $T < \infty$. Explain why a rational speculative asset price bubble could not exist in such an economy.

5. For assets have a finite horizon, its price cannot take the form of $P_t = f_t + b_t$ with $b_t \neq 0$ because at T $P_T = f_T = d_T$ which is the assent final dividend payment. We know that $b_T = 0$ then the bubble process $E_t[b_{t+1}] = \delta^{-1} b_t$ implying $E_{T-1}[b_T] = E_{T-1}[0] = \delta^{-1} b_{T-1}$, or $b_{T-1} = 0$. The arguments implies $b_t = 0$ for all previous dates $(t < T-1)$.