Macroeconomics

Lecture 1

A Recursive Problem

- Max $r_0(x_0,u_0)+r_1(x_1,u_1)+\dots.+r_T(x_T,u_T)+W_0(x_{T+1})$ (1.1)
- Subject to (i) x_0 given,
- and (ii) the transition equations

$$x_1 = g_0(x_0,u_0)$$
$$x_2 = g_1(x_1,u_1)$$
$$x_3 = g_2(x_2,u_2)$$
$$\vdots$$
$$x_{T+1} = g_T(x_T,u_T) \qquad (1.2)$$

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- Assume that:
- (1) there is a time separable function, $r_t(x_t,u_t)$, which is a concave function.

  Let x_t be an (nx1) vector of state variable at time t, t=0,1,…,T+1.

  Let u_t be a (kx1) vector of control variable at time t, t=0,1,…,T.

$$x_t = \begin{pmatrix} x_{1t} \\ x_{2t} \\ \vdots \\ x_{nt} \end{pmatrix}$$

$$u_t = \begin{pmatrix} u_{1t} \\ u_{2t} \\ \vdots \\ u_{kt} \end{pmatrix}$$

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- (2) there is a transition function whose variables are defined by the set $\{x_{t+1}, x_t, u_t : x_{t+1} \leq g_t(x_t, u_t), u_t \in R^K\}$ which is convex and compact set. (See definitions of these terms from Takayama, A . "Mathematical Economics" (1985), p.20 and p.32).
- Then

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- The values of $r_s(x_s, u_s)$ for $s \geq t$ are independent to the past values of u_v and x_v for $v < t$. This is the consequence of the time separable structure of (1.1) and (1.2)
- Forming the Lagrangian
- $$L = \sum_{t=0}^{T} r_t\left(x_t, u_t\right) + W_0\left(x_{T+1}\right) + \sum_{t=0}^{T} \lambda_t^{'}\left[g_t\left(x_t, u_t\right) - x_{t+1}\right] \quad (1.3)$$
- where λ_t is an (nx1) vector of Lagrange multipliers for $t = 0,1,\ldots,T$.

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- The 1st-order conditions

$$\frac{\partial L}{\partial u_t} = \frac{\partial r_t(x_t, u_t)}{\partial u_t} + \frac{\partial g_t(x_t, u_t)}{\partial u_t}\lambda_t = 0,\ t = 0, \ldots, T \quad \text{(1.4a)}$$

- $$\frac{\partial L}{\partial x_t} = \frac{\partial r_t(x_t, u_t)}{\partial x_t} + \frac{\partial g_t(x_t, u_t)}{\partial x_t}\lambda_t - \lambda_{t-1} = 0,\ t = 1, \ldots, T+1 \quad \text{(1.4b)}$$

- $$\frac{\partial L}{\partial x_{T+1}} = W_0^{'}(x_{T+1}) - \lambda_T = 0 \quad \text{(1.4c)}$$

$$x_{t+1} = g_t(x_t, u_t), \qquad t = 0, 1, \ldots, T. \quad \text{(1.4d)}$$

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- From (1.4b), one can have

$$\lambda_t = \frac{\partial r_{t+1}\left(x_{t+1}, u_{t+1}\right)}{\partial x_{t+1}} + \frac{\partial g_{t+1}\left(x_{t+1}, u_{t+1}\right)}{\partial x_{t+1}} \lambda_{t+1}$$

- Using this and (1.4c) recursively to eliminate λ_t , t=0,1,…,T, from (1.4a), we can have at t=0,1,…,T-1,

$$\frac{\partial r_t\left(x_t, u_t\right)}{\partial u_t} + \frac{\partial g_t\left(x_t, u_t\right)}{\partial u_t}\left\{\frac{\partial r_{t+1}}{\partial x_{t+1}} + \frac{\partial g_{t+1}}{\partial x_{t+1}}\left[\frac{\partial r_{t+2}}{\partial x_{t+2}} + \frac{\partial g_{t+2}}{\partial x_{t+2}}\left\{\frac{\partial r_{t+3}}{\partial x_{t+3}} + \frac{\partial g_{t+3}}{\partial x_{t+3}}\left\{\ldots + \frac{\partial g_T}{\partial x_T}\left[W_0^{'}\left(x_{T+1}\right)\right]\right\}\ldots\right\}\right]\right\} = 0$$

$$t = 0,1,\ldots,T-1$$

(1.5a)

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- $x_{t+1} = g_t\left(x_t, u_t\right), \qquad t = 0,1,\ldots,T.$ (1.5b)

- At t =T and from using (1.4a) and (1.4d), we have that

- $$\frac{\partial r_T\left(x_T, u_T\right)}{\partial u_T} + \frac{\partial g_T\left(x_T, u_T\right)}{\partial u_T} W_0^{'}\left(x_{T+1}\right) = 0$$ (1.5c)

- $$x_{T+1} = g_T\left(x_T, u_T\right)$$ (1.5d)

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- Now, consider the following optimization problem at T

$$\max \quad r_T(x_T, u_T) + W_0(x_{T+1}) \qquad (1.1a)$$

$$s.t. \quad x_T \quad given,$$

$$x_{T+1} = g_T(x_T, u_T) \qquad (1.2a)$$

- It should be pointed out that this problem is a sub-problem of problem (1.1) and (1.2)

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- And the 1st-order conditions are
- $$\frac{\partial L}{\partial u_T} = \frac{\partial r_T\left(x_T, u_T\right)}{\partial u_T} + \frac{\partial g_T\left(x_T, u_T\right)}{\partial u_T}\lambda_T = 0, \quad (1.4aa)$$
- $$\frac{\partial L}{\partial x_T} = \frac{\partial r_T\left(x_T, u_T\right)}{\partial x_T} + \frac{\partial g_T\left(x_T, u_T\right)}{\partial x_T}\lambda_T - \lambda_{T-1} = 0, \quad (1.4bb)$$
- $$\frac{\partial L}{\partial x_{T+1}} = W_0^{'}\left(x_{T+1}\right) - \lambda_T = 0 \quad (1.4c)$$
-
- $$x_{T+1} = g_T\left(x_T, u_T\right), \quad (1.4dd)$$

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- By using (1.4aa) and (1.4c), we have

$$\frac{\partial r_T\left(x_T, u_T\right)}{\partial u_T}+\frac{\partial g_T\left(x_T, u_T\right)}{\partial u_T} W_0^{'}\left(x_{T+1}\right)=0 \qquad (1.5cc)$$

And recall (1.4dd),

$$x_{T+1}=g_T\left(x_T, u_T\right) \qquad (1.5d)$$

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- Given x_T , equations (1.5cc) and (1.5d) form a system of (n+k) equations in (x_{T+1}, u_T).
- We solve these equations for x_{T+1} and u_T as functions of x_T,
- $x_{T+1} = f_T(x_T)$, $u_T = h_T(x_T)$
- Where $f_T(x_T) \equiv g_T[x_T, h_T(x_T)]$.

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- By repeating the same process again for t =T-1, the optimization problem at T-1 is

$$\max \quad r_{T-1}\left(x_{T-1},u_{T-1}\right)+r_T\left(x_T,u_T\right)+W_0\left(x_{T+1}\right), \qquad (1.1aa)$$

$$s.t. \qquad x_{T-1} \quad given,$$

$$\left.\begin{aligned} x_T &= g_{T-1}\left(x_{T-1},u_{T-1}\right), \\ x_{T+1} &= g_T\left(x_T,u_T\right). \end{aligned}\right\} \qquad (1.2aa)$$

- This problem is, again, a sub-problem of problem (1.1) and (1.2)

A Recursive Problem

- And the 1st-order conditions are
- $$\frac{\partial L}{\partial u_t} = \frac{\partial r_t\left(x_t, u_t\right)}{\partial u_t} + \frac{\partial g_t\left(x_t, u_t\right)}{\partial u_t}\lambda_t = 0, \quad t = T-1, T.$$ (1.4aaa)
- $$\frac{\partial L}{\partial x_t} = \frac{\partial r_t\left(x_t, u_t\right)}{\partial x_t} + \frac{\partial g_t\left(x_t, u_t\right)}{\partial x_t}\lambda_t - \lambda_{t-1} = 0, \quad t = T-1, T.$$ (1.4bbb)
- $$\frac{\partial L}{\partial x_{T+1}} = W_0^{'}\left(x_{T+1}\right) - \lambda_T = 0$$ (1.4c)
-
- $$x_{t+1} = g_t\left(x_t, u_t\right), \quad t = T-1, T.$$ (1.4ddd)

A Recursive Problem

- By using (1.4aaa),(1.4bbb) and (1.4c), we have, at t=T-1, that

$$\frac{\partial r_{T-1}\left(x_{T-1},u_{T-1}\right)}{\partial u_{T-1}}+\frac{\partial g_{T-1}\left(x_{T-1},u_{T-1}\right)}{\partial u_{T-1}}\left[\frac{\partial r_{T}\left(x_{T},u_{T}\right)}{\partial x_{T}}+\frac{\partial g_{T}\left(x_{T},u_{T}\right)}{\partial x_{T}}W_0^{'}\left(x_{T+1}\right)\right]=0$$

(1.5ccc)

Equation (1.5ccc) can also be obtained from (1.5a), given that t=T-1. Then recall (1.4ddd)

$$x_{t+1}=g_t\left(x_t,u_t\right),\quad t=T-1.$$

(1.5ddd)

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- Next, use $h_T(x_T)$ and $g_T(x_T,h_T(x_T))$ to replace u_T and X_{T+1} in (1.5ccc), we then have

$$\frac{\partial r_{T-1}\left(x_{T-1},u_{T-1}\right)}{\partial u_{T-1}} + \frac{\partial g_{T-1}\left(x_{T-1},u_{T-1}\right)}{\partial u_{T-1}}\left[\frac{\partial r_T\left(x_T,h_T\left(x_T\right)\right)}{\partial x_T} + \frac{\partial g_T\left(x_T,h_T\left(x_T\right)\right)}{\partial x_T} W_0^{'}\left(g_T\left(x_T,h_T\left(x_T\right)\right)\right)\right] = 0 \qquad (1.5ccc*)$$

$$and, \quad x_T = g_{T-1}\left(x_{T-1},u_{T-1}\right), \qquad (1.5ddd*)$$

A Recursive Problem

- Given x_{T-1} , equations (1.5ccc*) and equations (1.5ddd*) form a system of (n+k) equations in (x_T, u_{T-1}).
- Solving these (n+k) equations for x_T and u_{T-1} as functions of x_{T-1}:
- $x_T = f_{T-1}(x_{T-1})$,
- $u_{T-1} = h_{T-1}(x_{T-1})$