

EE320 (1/2015)

INTRODUCTORY MATHEMATICAL ECONOMICS

OPTIMIZATION WITHOUT CONSTRAINTS:

MORE-THAN-ONE-INDEPENDENT VARIABLE CASES

Topics

- The differential version of optimization condition:
One variable case
- Two choice variable optimization
 - Conditions for maximum or minimum
 - Economics examples
- Multivariable optimization
 - Conditions for maximum or minimum
 - Economics examples

Differential Version of Optimization Conditions: One Variable Case

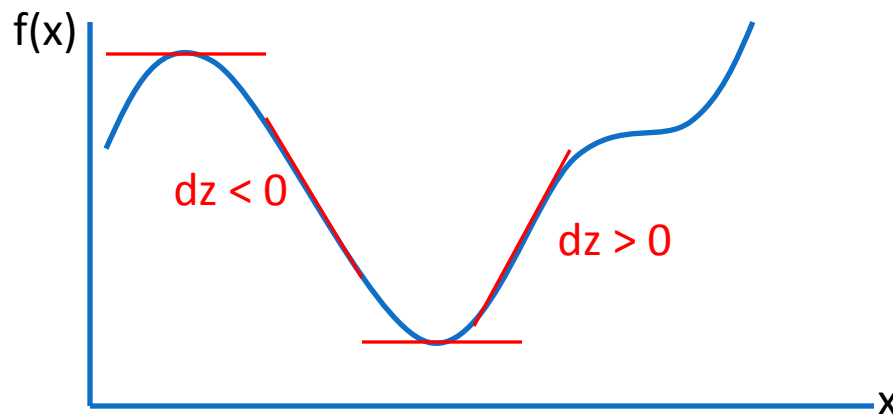
- Given $z = f(x)$, the differential of z is:

$$dz = f'(x)dx, \quad \text{where } f'(x) = dz/dx.$$

- The first-order condition (in terms of derivative) for an optimum:

$$f'(x) = 0$$

- Graph



- The first-order differential condition for any arbitrary $dx \neq 0$:

$$dz = 0.$$

Differential Version of Optimization Conditions: One Variable Case

- The **second-order sufficient condition** (in terms of derivative):
 - For a maximum of z : $f''(x) < 0$.
 - For a minimum of z : $f''(x) > 0$.

- Rewrite the S.O.C. in terms of differential:

$$d^2z = d(dz) = d[f'(x)dx] = [df'(x)]dx = [f''(x)dx]dx$$

$$\rightarrow d^2z =$$

- The **second-order differential condition**:
 - For a maximum of z : _____.
 - For a minimum of z : _____.

Example: 1-variable optimization

- Example: $y = 10x - x^2$

TWO CHOICE VARIABLE OPTIMIZATION

First-Order Condition

- For $z = f(x, y)$, the first-order necessary condition is:
 $dz = 0$ for arbitrary values of dx and dy , not both zero.
- The total differential is:

$$dz = f_x dx + f_y dy$$

$$dz = 0 \rightarrow \underline{\hspace{2cm}}$$

Theorem: A differentiable function $z = f(x, y)$ can only have a maximum or minimum at an interior point (x_0, y_0) if it is a stationary point. That is, if the point $(x, y) = (x_0, y_0)$ satisfies the two F.O.C. equations:

$$f_x(x_0, y_0) = 0 \quad \text{and} \quad f_y(x_0, y_0) = 0.$$

Example: F.O.C. for Two-Variable Case

Example: $z = f(x, y) = 10x + 10y + xy - x^2 - y^2$

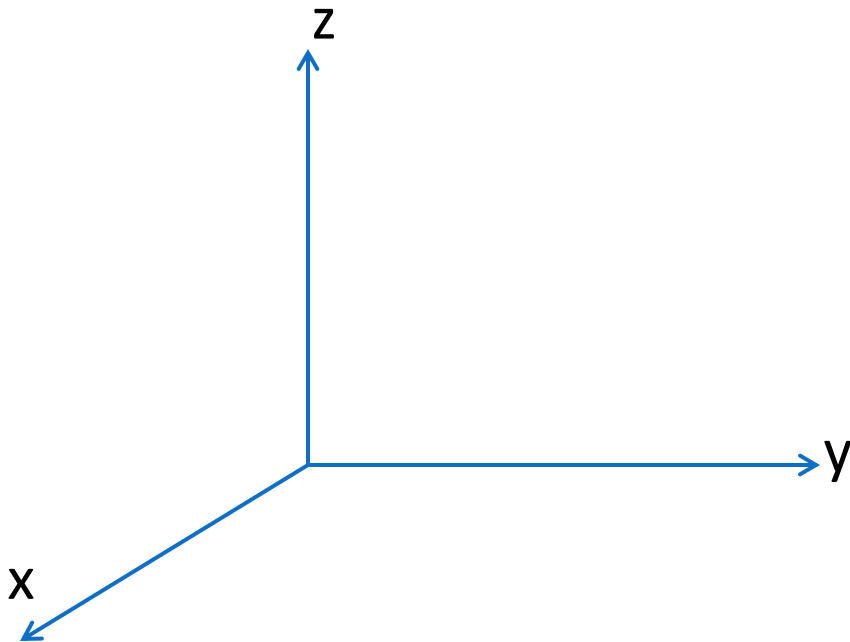
➤ F.O.C.'s are:

➤ Critical points are:

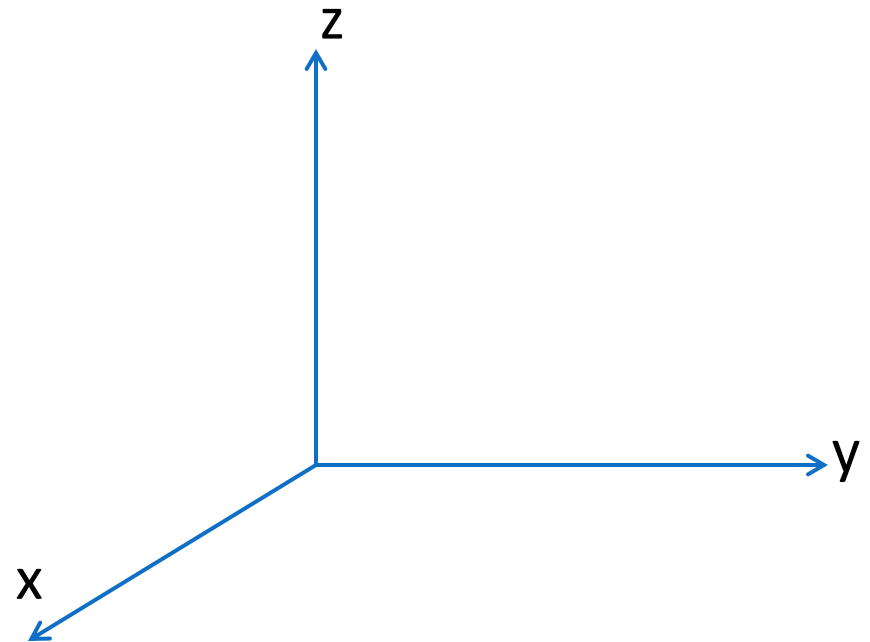
Extreme Values of a Function of Two Variables:

$$z = f(x, y)$$

Maximum value

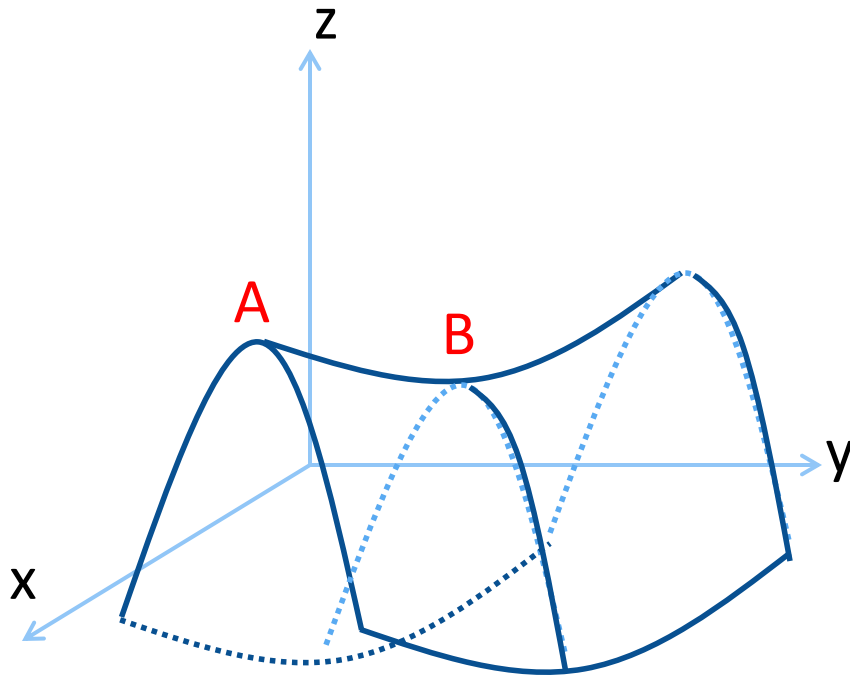


Minimum value

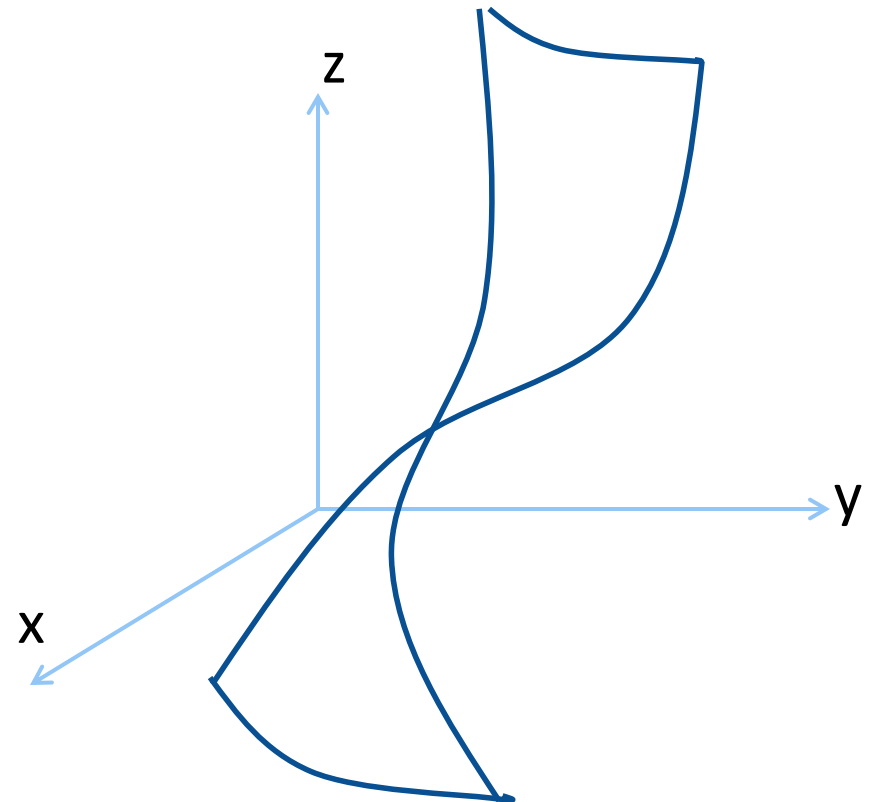


The FOC is necessary but *not sufficient* condition.

Saddle Point



Inflection Point



Second-Order Conditions

- To test for a maximum or a minimum, we need to check the second-order partial derivatives.
- Second-Order Total Differentials:

$$d^2z = d(dz)$$

Hessian Matrix of Second Partial

- The quadratic form $d^2z = f_{xx}dx^2 + 2f_{xy}dxdy + f_{yy}dy^2$ can be written in matrix form as:
- The matrix in which the second-order partial derivatives are the elements is the **Hessian matrix**, and its determinant is called a “**Hessian determinant**”.
- First leading principal minor: $|H_1| =$
- Second leading principal minor: $|H_2| =$

Rules for Maximum and Minimum: 2-Variable Case*

- Given that the first-order necessary condition is satisfied and , the **second-order sufficient condition** for $z = f(x, y)$ is

1. Maximum:

$$d^2z < 0 \quad \text{iff } f_{xx} < 0, f_{yy} < 0, \text{ and } f_{xx}f_{yy} > (f_{xy})^2$$

2. Minimum:

$$d^2z > 0 \quad \text{iff } f_{xx} > 0, f_{yy} > 0, \text{ and } f_{xx}f_{yy} > (f_{xy})^2$$

3. Otherwise, we have a saddle point.

Note: The second-order partial derivatives are to be evaluated at the stationary point where $f_x = f_y = 0$.

*See additional notes on the restrictions for the signs of f_{xx} , f_{xy} , and f_{yy} .

Summary: Necessary and Sufficient Conditions for a Maximum and a Minimum

Given that $z = f(x, y)$,

Condition	Maximum	Minimum
First-order necessary	$f_x = f_y = 0$	$f_x = f_y = 0$
Second-order sufficient	$f_{xx} < 0,$ $f_{yy} < 0,$ <i>and</i> $f_{xx}f_{yy} > (f_{xy})^2$	$f_{xx} > 0,$ $f_{yy} > 0,$ <i>and</i> $f_{xx}f_{yy} > (f_{xy})^2$

Example: Second-Order Condition

- $z = f(x, y) = 10x + 10y + xy - x^2 - y^2$

FOC:

SOC:

➤ $f_{xx} =$

➤ $f_{xy} =$

➤ $f_{yy} =$

➤ $f_{yx} =$

➔ d^2z

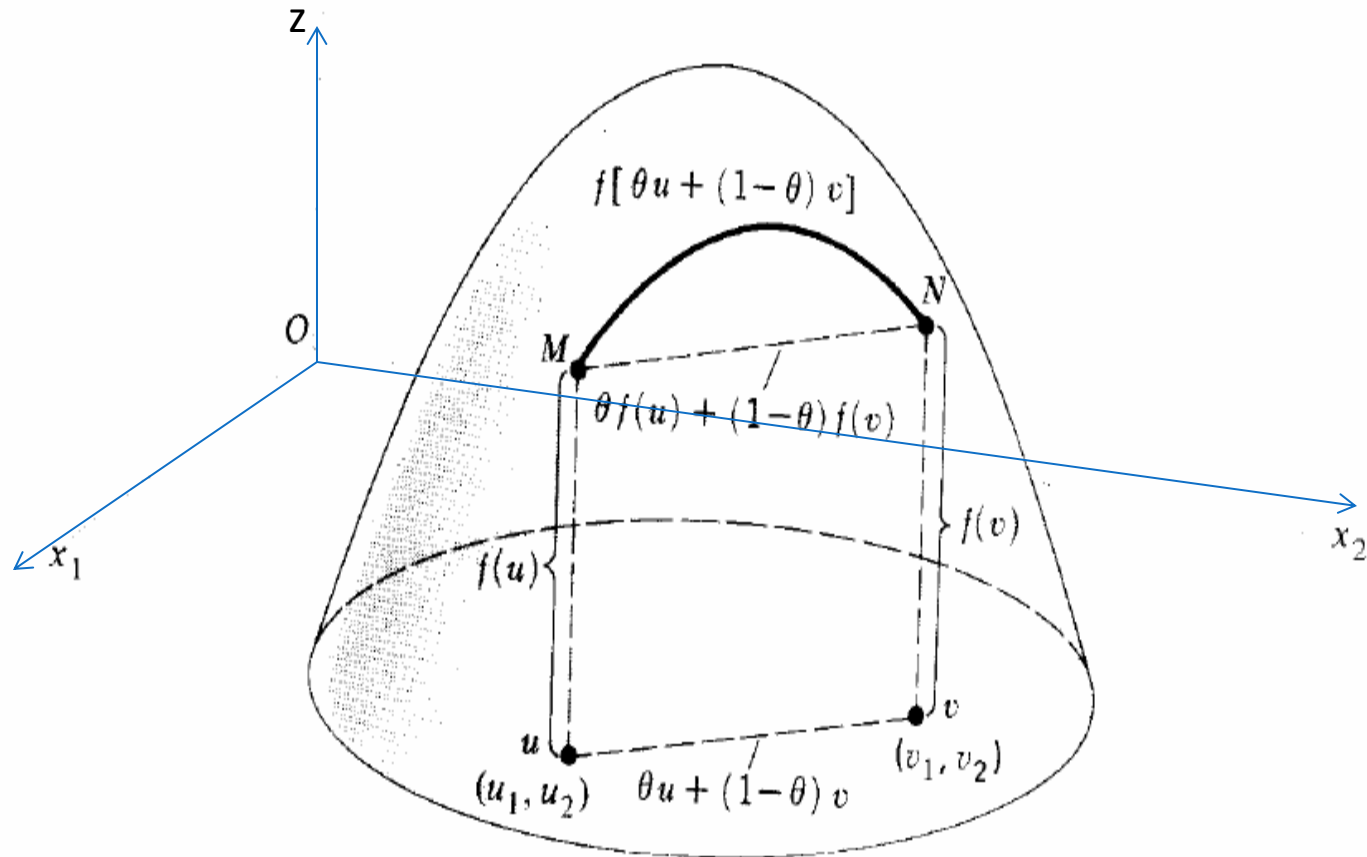
Example: 2-choice variable optimization

- $z = f(x, y) = -2x^2 - 2xy - 2y^2 + 36x + 42y - 158.$

Concavity and Convexity: 2 variable case

- Consider a two-variable function $z = f(x_1, x_2)$.
- **Definition:**
 - The function is *concave* (*strictly concave*) if and only if, for any pair of distinct points M and N on its graph – surface – line segment MN lies either *on or below* (*entirely below*) the surface.
 - The function *convex* (*strictly convex*) if and only if, for any pair of distinct points M and N on its graph – surface – line segment MN lies either *on or above* (*entirely above*) the surface.

Strictly Concave Function



A function f is **strictly concave** (**convex**) iff, for any pair of distinct points u and v in the domain of f , and for $0 < \theta < 1$,

$$\theta f(u) + (1 - \theta)f(v) < (>) f[\theta(u) + (1 - \theta)(v)]$$

Height of the line MN

Height of arc

Checking Concavity and Convexity by the Derivative Conditions

- Consider a function $z = f(x_1, x_2)$, which is twice continuously differentiable. For such a function, second-order partial derivatives exist, and thus d^2z is defined.
- Concavity and convexity can then be checked by the **sign of d^2z** :
 - $z = f(x_1, x_2)$ is **strictly concave** if, and only if, d^2z is everywhere ***definite negative*** ($d^2z < 0$).
 - ➔ z^* is a maximum.
 - $z = f(x_1, x_2)$ is **strictly convex** if, and only if, d^2z is everywhere ***definite positive*** ($d^2z > 0$).
 - ➔ z^* is a minimum.

where z^* is the value of function z evaluated at the stationary point where $f_x = f_y = 0$.

Examples: Convexity and Concavity

Check the following functions for concavity and convexity by the derivative conditions.

- Example 1: $z = 2x^2 - xy + y^2$

- Example 2: $z = x_1^2 + x_2^2$

Application: Duopoly

- There are two firms, with identical cost: $TC_i = cQ_i$, $i = 1, 2$, and market demand is $P = a - bQ$, $Q = Q_1 + Q_2$.
- Find Q_1, Q_2 that maximizes each firm's profit.

Application: Multiproduct Firm (1)

- Competitive firm

Let $p_1 = 6$, $p_2 = 9$ and $TC = 2Q_1^2 + Q_1Q_2 + 2Q_2^2$.

Find Q_1 and Q_2 that maximize the firm's profit.

Application : Multiproduct Firm (2)

- Monopolist

Let $p_1 = 35 - Q_1$, $p_2 = 33 - Q_2$ and $TC = 2Q_1^2 + Q_1Q_2 + 2Q_2^2$.

Find Q_1 and Q_2 that maximize the firm's profit.

Application : Multiplant Firm

- Let $P = 25$, $TC_1 = 2Q_1^2 + 5Q_1 + 10$, and $TC_2 = 2Q_2^2 + 3Q_2 + 15$.
Find Q_1 and Q_2 that maximize the firm's profit.

Application :

Multimarket Monopoly (Price Discrimination)

- Let $R = R_1(Q_1) + R_2(Q_2)$ and $C = C(Q)$ where $Q = Q_1 + Q_2$. Find the FONC and SOSC for profit maximization.

Application : Price Discrimination

- Let $p_1 = 22 - 2Q_1$, $p_2 = 10 - 0.5Q_2$ and $TC = 2Q + 5$.
Find Q_1 , Q_2 , p_1 , and p_2 that maximize the firm's profit.
- Max $\pi = (22 - 2Q_1)Q_1 + (10 - 0.5Q_2)Q_2 - [2(Q_1 + Q_2) + 5]$
- FONC: $\pi_1 = \underline{\hspace{2cm}}$
 $\pi_2 = \underline{\hspace{2cm}}$
 ➔ $Q^*_1 = \underline{\hspace{1cm}}, Q^*_2 = \underline{\hspace{1cm}}$
- SOSC:
- ➔ SOSC for max are satisfied.

Application : Input Decision of a firm

- Let $Q = f(K, L) = 5K^{0.5}L^{0.25}$, $P = 4$, $w = 10$, $r = 5$. Find K^* and L^* that maximize profit.

Comparative-Static Aspects of Optimization

Example: Consider a two-product firm in perfect competition.

The firm's revenue is $R = P_{10}Q_1 + P_{20}Q_2$ (P_{10} and P_{20} are exogenous)

And the cost function is $C = 2Q_1^2 + Q_1Q_2 + 2Q_2^2$.

→ **Reduced-form solutions** are the optimal output levels expressed in terms of exogenous variables (P_{10} and P_{20}):

→ **Comparative-statics of the model:**

$$\frac{\partial Q_1^*}{\partial P_{10}} = \frac{4}{15} \quad ; \quad \frac{\partial Q_1^*}{\partial P_{20}} = -\frac{1}{15} \quad ; \quad \frac{\partial Q_2^*}{\partial P_{10}} = -\frac{1}{15} \quad ; \quad \frac{\partial Q_2^*}{\partial P_{20}} = \frac{4}{15}$$

Comparative-Static Aspects of Optimization

- General-Function Models

Consider the input-decision problem. Given that $R = P \cdot Q(K, L)$ and $C = wL + rK$, where P , w , and r are exogenous, find dL^*/dP and dK^*/dP .

MULTIVARIABLE OPTIMIZATION

3-Variables Optimization

- Let $z = f(x_1, x_2, x_3)$. Total differential is:

$$dz =$$

- First-order necessary condition for an extremum of z is:

- Second-order total differential:

$$d^2z =$$

- ➔ Hessian matrix:

3-Variables Optimization

- The leading principal minors of the Hessian matrix are:
- Thus, **second-order sufficient condition** for an extremum of z :
 1. z^* is a *minimum* if $|H_1| > 0$; $|H_2| > 0$; $|H_3| > 0$
(i.e. d^2z is *positive definite*.)
 2. z^* is a *maximum* if $|H_1| < 0$; $|H_2| > 0$; $|H_3| < 0$
(i.e. d^2z is *negative definite*.)

Note: All the leading principal minors are evaluated at the stationary points where $f_1 = f_2 = f_3 = 0$.

n -variable Optimization

- Let $z = f(x_1, x_2, \dots, x_n)$.
- Total differential:

➤ F.O.N.C.:

➤ S.O.S.C:

Hessian matrix:
$$H = \begin{bmatrix} f_{11} & f_{12} & \cdots & f_{1n} \\ f_{21} & f_{22} & \cdots & f_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ f_{n1} & f_{n2} & \cdots & f_{nn} \end{bmatrix}$$

where $|H_1|, |H_2|, \dots, |H_n|$ are the leading principal minors.

- For a *maximum* in z , all the principal minors *alternate in sign, the first one being negative*:
- For a *minimum* in z , all the principal minors must be *positive*:

Necessary and Sufficient Conditions for a Maximum and Minimum (n - choice variables)

Condition	Maximum	Minimum
First-order necessary	$f_1 = f_2 = \dots = f_n = 0$	$f_1 = f_2 = \dots = f_n = 0$
Second-order sufficient	$ H_1 < 0, H_2 > 0,$ $ H_3 < 0, \dots$ or $(-1)^i H_i > 0$ for $i = 1, 2, \dots, n$	$ H_1 > 0, H_2 > 0,$ $ H_3 > 0, \dots$ Or $ H_i > 0$ for $i = 1, 2, \dots, n$

Application: Multimarket Monopoly

- Let $p_1 = 63 - 4Q_1$, $p_2 = 105 - 5Q_2$, $p_3 = 75 - 6Q_3$ and $TC = 15Q + 20$. Find Q_i and p_i , for $i = 1, 2, 3$, that maximize the firm's profit.