Optimization theory: *single variable Calculus*

EC320

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Optimization theory in general.

- A mathematical field combining together three different strands of mathematical tools:
  - Matrix algebra
  - Function/relation
  - Calculus
- Developed to solve for the solution of the problem that can be mathematically characterized in the following form

$$\max(mim)_x \;\; f(x) \;, \qquad x \in D.$$

Optimization theory in general.

- By the nature of problem, it has been extensively applied into so many engineering problems.
- Economics have started applying this tool in economics problem lately.

Optimization in economics: why?

- We've seen before that equilibrium economic models include several behavioral equations that captures behavior of groups/persons engaged in the model.
- All these behavioral equations are actually derived from *the first principle* that agents are rational and choose for optimal decision, resulting in the behavior that can be captured by the proposed function.

Optimization in economics

- Market demand model
  - Demand equation is the result of maximizing utility, subject to budget set.
  - Supply equation is the result of profit-maximizing decision of firm.
- National income model
  - Aggregate consumption function is derived the aggregate of individual consumption function which is the result from intertemporal consumption problem.

Optimization in economics

- Optimization can help economists getting into deeper foundations at the level of individual decision marking.
- With this accurate knowledge of how people make decisions, having aggregated up these decision to generate aggregate behaviors, the model developed should hypothetically result in better prediction of how economy works than the one with ad-hoc assumption used.

Plan to go

- BASIC Calculus
  - Limit/continuity
  - Derivative
  - Higher order derivative
- Single variable optimization
- Multivariate calculus and optimization (after midterm)

Calculus

- Limit & Continuity: review yourself!
- We will not talk about this.
  - Left/right limit
  - L'Hospital rule

Derivative: general concept

- Definition

$$f'(x) = \lim_{h \to 0} \frac{f(x_0+h)-f(x_0)}{h} = \frac{dy}{dx} = \frac{df(x)}{dx}$$

- Derivative is used to *approximate* the value of slope of a function at a given point x_0, when change in X (by h units) is very small.

Derivative and Slope in graphical illustration.

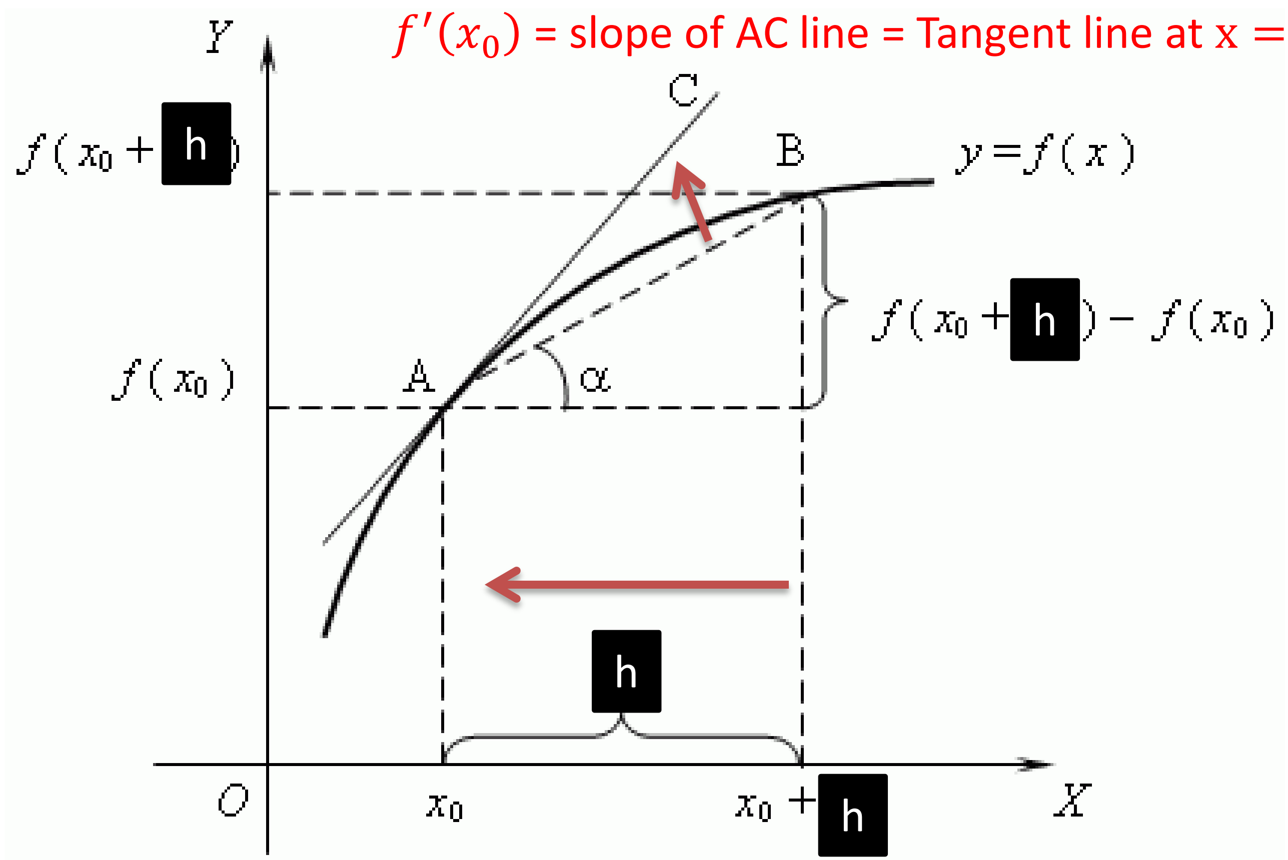
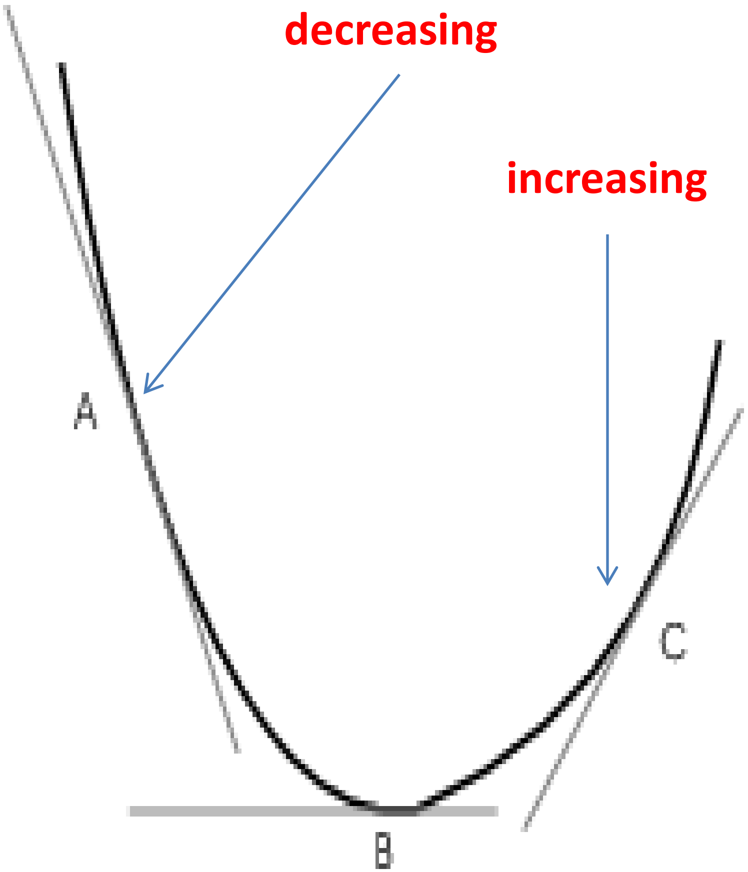


Fig. 1

Derivative: increasing v.s. decreasing function



- A positive slope means the function is increasing at that point whereas a negative slope means the function is decreasing.

Derivative: formula

- Derivative of some commonly used functions in economics

	f(x)	f'(x)
1	c	0
2	Ax^n	Anx^{n-1}
3	Ae^x	Ae^x
4	lnx	$\frac{1}{x}$
5	a^x	$a^x lna$
6	$log_a x$	$\frac{1}{xlna}$

Derivative: higher order

- Higher order derivative.

  - 2nd order ➔ $f''(x) = \frac{df'(x)}{dx} = \frac{d^2 f(x)}{dx^2}$

  - 3rd order ➔ $f^3(x) = \frac{df''(x)}{dx} = \frac{d^3 f(x)}{dx^3}$

  - nth order ➔ $f^n(x) = \frac{df^{n-1}(x)}{dx} = \frac{d^n f(x)}{dx^n}$

Derivative

- Rule of differentiation
  - Summation rule: $h(x) = f(x) \pm g(x)$

    $h'(x) = f'(x) \pm g'(x)$

  - Product rule $\mathrm{y = f(x)g(x)}$ $\quad \frac{dy}{dx} = f(x)g'(x) + g(x)f'(x)$

. $y = (x^2+1)(x^4+4x+1)$

Derivative

- Quotient rule

$$y = \frac{f(x)}{g(x)} \qquad \frac{dy}{dx} = \frac{g(x)f'(x) - f(x)g'(x)}{[g(x)]^2}$$

$$\text{ex. } y = \frac{x^2+2x+3}{x^3+7}$$

Derivative

- Chain rule:

$$y = (f \circ g)(x) \Rightarrow \frac{dy}{dx} = f'\big(g(x)\big)g'(x)$$

$$\text{ex. } y = ln(x^2 + 1)$$

$$u = x^2 + 1$$

Derivative

ex. $y = ln(x^2+1)^5$

$u = (x^2+1)^5$

Example 1

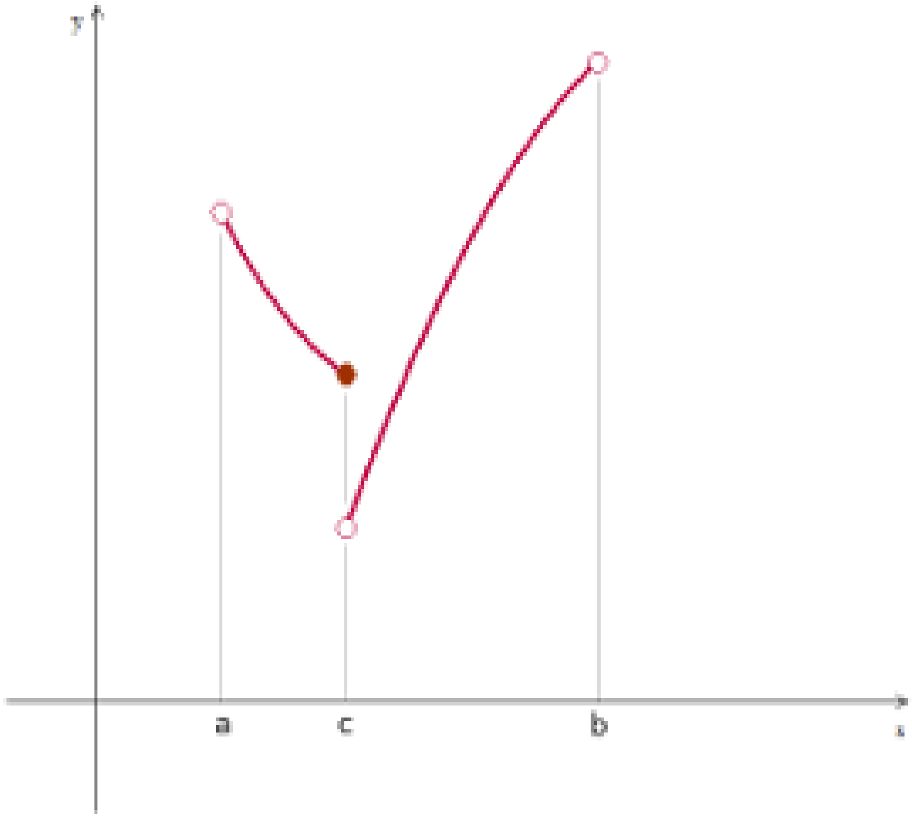
- $C = C(Y)$
- $C = 10 + 0.1Y + 6\sqrt{Y}$
- Find the value of MPC when Y = 100 and 1000.

Non-differentiable function

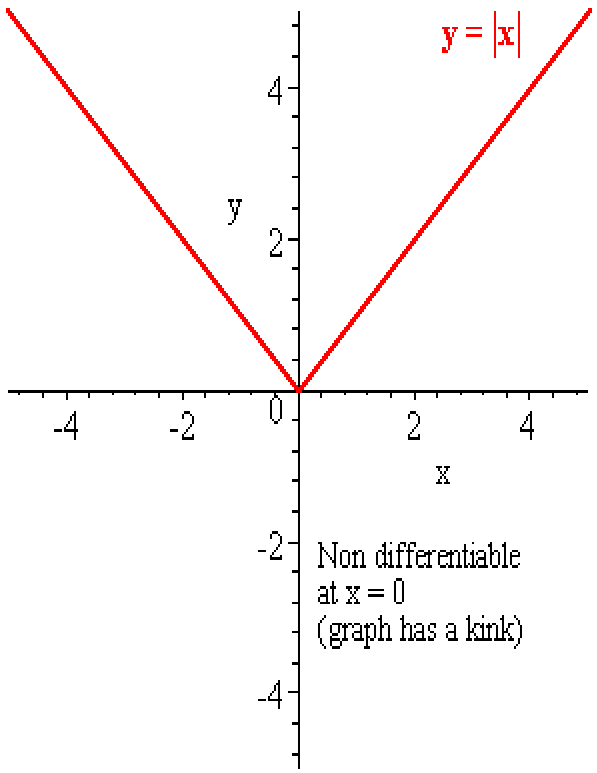
- At some points of domain of a given function, derivative may not exist.
- For two possible reasons,
  - Not continuity
  - Not smooth
- Mathematically, we call a function as a differentiable function if derivative exists for all the domain set defining that function.

Non-differentiable function

Not continuity



Not smooth ➔ non-differentiable.



Some economic Applications

- Revenue function
- Production function
- Cost function

Application: revenue function

- Total revenue (TR), Average revenue (AR), marginal revenue (MR) and elasticity of demand.

Application: REVENUE

- TR = P*Q

- AR = TR/Q = P = Demand curve

- MR?

$$\mathrm{TR} = \mathrm{P(Q)}\cdot \mathrm{Q} \quad , \quad \mathrm{AR} = [\mathrm{P(Q)\text{-}Q}]/\mathrm{Q} = \mathrm{P(Q)}$$

$$MR = \frac{dTR}{dQ} \boxed{}$$

Application: REVENUE

- Suppose that $P = P(Q)$ as a demand function.

$P = 20 - 5Q$ (Q: quantity and P: price per unit.)

TR =

MR =

Application: REVENUE

$$\mathrm{TR} = \mathrm{P(Q)} \cdot \mathrm{Q} \quad , \quad \mathrm{AR} = [\mathrm{P(Q)\text{-}Q}]/\mathrm{Q} = \mathrm{P(Q)}$$

$$MR = \frac{dTR}{dQ} = \underline{QP'(Q) + P(Q)\frac{dQ}{dQ}} = P(Q) + Q\frac{dP(Q)}{dQ}$$

By product rule

Application: REVENUE

- Marginal revenue and *Elasticity*

$$
\begin{aligned}
P(Q) + Q\frac{dP(Q)}{dQ} &= AR + Q \cdot \frac{dP(Q)}{dQ} \times \frac{P(Q)}{P(Q)} \\
&= AR + \frac{Q}{P(Q)} \cdot \frac{dP(Q)}{dQ} \cdot P(Q) \\
&= AR + \frac{P(Q)}{\varepsilon_p^d} \\
MR &= AR\left(1 + \frac{1}{\varepsilon_p^d}\right)
\end{aligned}
$$

Don't forget that $\varepsilon_p^d < 0$.

$\left|\varepsilon_p^d\right| < 1$ Inelastic

$\left|\varepsilon_p^d\right| = 1$ Unit Elastic

$\left|\varepsilon_p^d\right| > 1$ Elastic

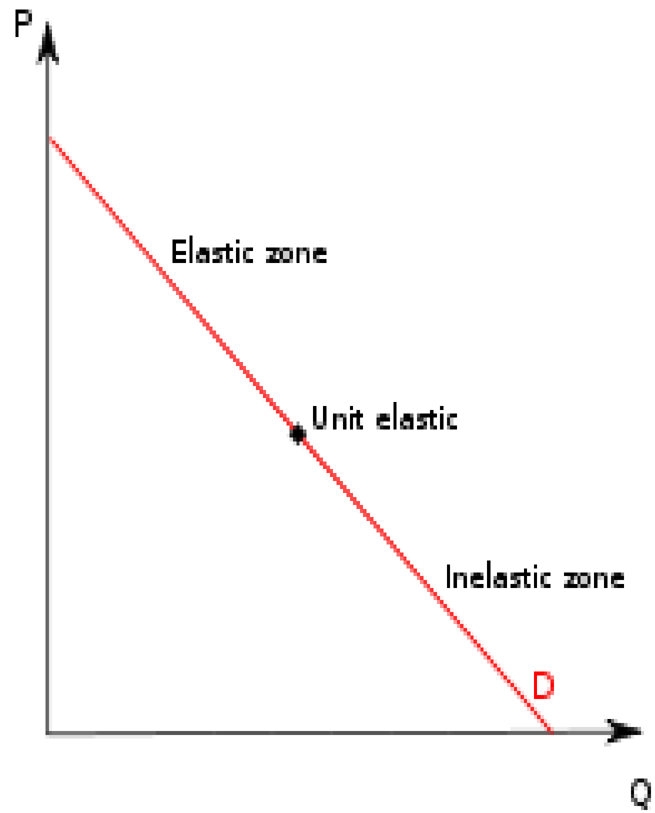
If $\varepsilon_p^d < -1$ $\left(\left|\varepsilon_p^d\right| > 1\right)$ ➔ $1 + \frac{1}{\varepsilon_p^d} > 0$. ➔ MR > 0. That is, TR increases as Q increases.

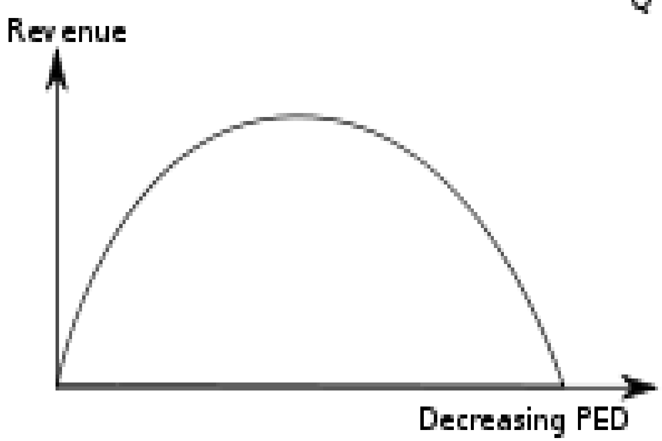
Under which case of price changes, would the quantity increase?

Lower the price is good way to boost total revenue.

Application: REVENUE

- Relationship between price setting and the change in the total revenue can be analyzed by using the above equation.
- Can the increase in price boost total revenue?
  - The equation tells us that it depends on the value of elasticity.
  - Yes, if the demand is inelastic, a decrease in the quantity would increase the total revenue. A decrease in quantity is associated with an increase in price. So, this works!
  - On the other hand, if demand is elastic, an increase in price would cause a total revenue to drop.





- **Linear case:**
  - Mid-point is where demand has unit elastic.
  - As we have shown before that absolute value of elasticity of demand decreases as Q increases, above the mid-point is elastic region.
  - Thus lowering price is good way.
  - Below the mid-point are region of inelastic demand.

Application: REVENUE

- What if $P = kQ^{-b},\ k > 0\ and\ b > 0$.

Elasticity:

Marginal revenue:

Application: PRODUCTION FUNCTION

- Production function: Y = f(K,L)
- K is fixed. ➔ Y = f(L).
- Average product $AP = \frac{f(L)}{L}$
- Marginal product? $MP = \frac{df(L)}{dL} = f'(L)$

Application: PRODUCTION FUNCTION

- Relationship between AP and MP.

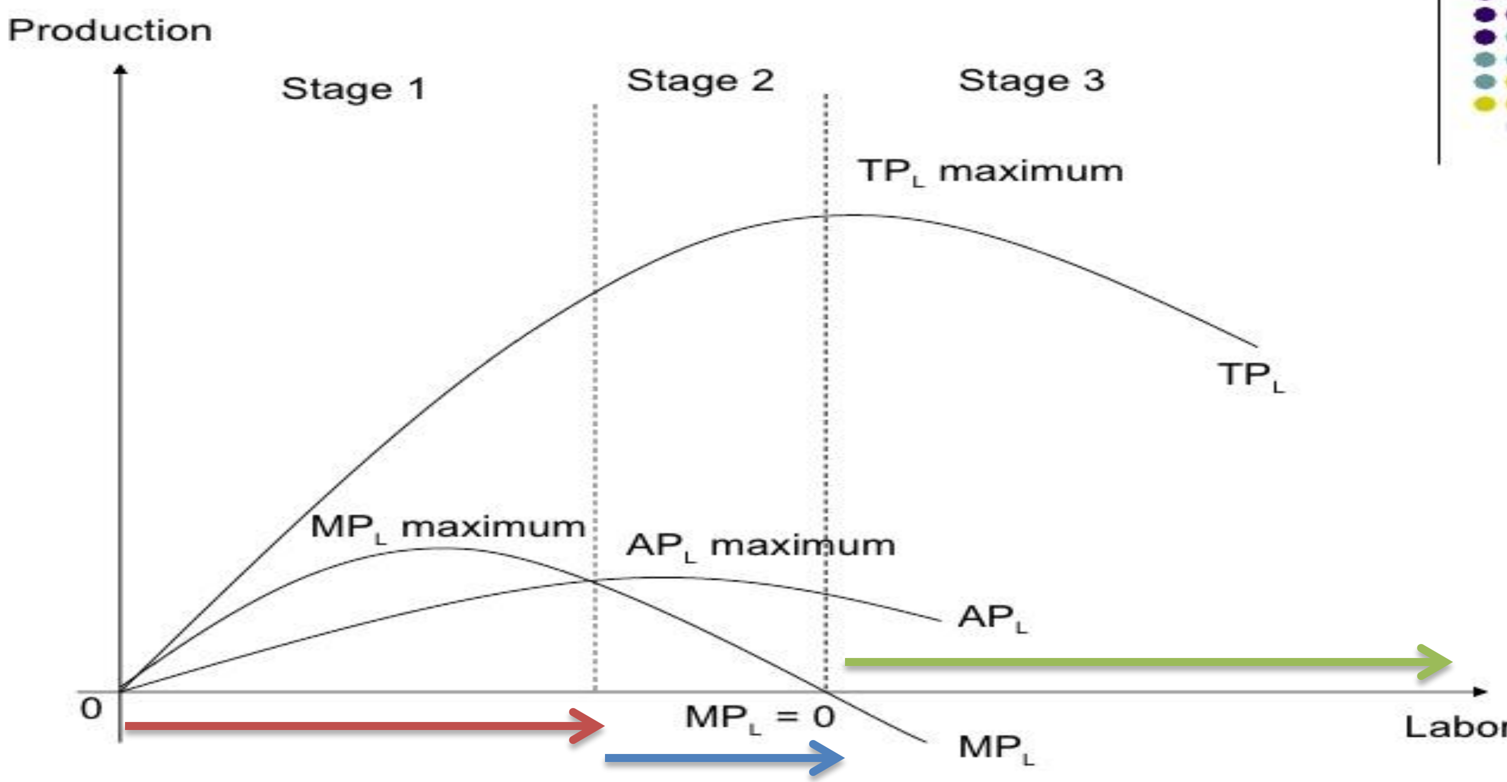
$$\frac{dAP}{dL} = \frac{Lf'(L) - f(L)\frac{dL}{dL}}{L^2} = \frac{MP}{L} - \frac{AP}{L} = \frac{MP - AP}{L}$$

$$I)\ MP > AP \quad \Rightarrow \quad \frac{dAP}{dL} > 0 \longrightarrow$$ AP is rising

$$II)\ MP = AP \quad \Rightarrow \quad \frac{dAP}{dL} = 0$$

$$III)\ MP < AP \quad \Rightarrow \quad \frac{dAP}{dL} < 0 \longrightarrow$$ AP is decreasing

Application: PRODUCTION FUNCTION



3 Stages of Production Curve in Short Run

Example

- CES: $y = [(1-\alpha)K^{\rho} + \alpha L^{\rho}]^{\frac{1}{\rho}}$ where $\rho > 0$.

If K = 1

AP =

AP' =

MP =

Application: COST FUNCTION

- Cost function
  - Generally speaking, TC = TFC + TVC(Q).
  - For example, $C(Q) = 2 + 3Q + Q^2$
    - Fixed cost = ________________________, TC(Q=0)
    - TVC = _________________________________, TVC(Q=0) = 0

- Average total cost $\mathrm{AC} = \frac{C(Q)}{Q}$

- Marginal cost $MC = \frac{d}{dQ}C(Q)$

Application: COST FUNCTION

- Relationship between AC and MC.

$$\frac{dAC}{dQ} = \frac{QC'(Q) - C(Q)\frac{dQ}{dQ}}{Q^2} = \frac{MC}{Q} - \frac{AC}{Q} = \frac{MC - AC}{Q}$$

$$I) MC < AC \rightarrow \frac{dAC}{dQ} < 0$$

$$II) MC = AC \rightarrow \frac{dAC}{dQ} = 0$$

$$III) MC > AC \rightarrow \frac{dAC}{dQ} > 0$$

Example

- $TC = \frac{5}{q+1} + 5 + 5q + q^2$

Fixed cost =

Total Variable Cost =

Average fixed cost =

Marginal cost =

Rising cost →