B.E. International Program

Faculty of Economics, Thammasat University

EE 320 Introductory Mathematical Economics

Semester 2/2013

Practice Problem 7

(Derivatives of More-Than-One Independent Variable Function)

Suggested Answers

1. Suppose that Mr. A's utility depends on two commodities: x_1 and x_2. His utility function is given by

$$U = 25x_1 + 27x_2 - 3x_1^2 - 7x_1x_2 - 4x_2^2$$

(a) Determine the marginal utility function of each commodity.

Ans. $MU_1 = 25 - 6x_1 - 7x_2$

$MU_2 = 27 - 7x_1 - 8x_2$

(b) Find the marginal rate of substitution (MRS) between the two commodities when $x_1 = 2$ and $x_2 = 1$.

Ans. $MRS_{12} = \frac{dx_2}{dx_1} = -\frac{MU_1}{MU_2} = -1.2$

2. Write the Hessian matrix for each of the following functions:

(a) $U(x,y) = 6x^2 + 5xy + 9y^2$

Ans. $H = \begin{bmatrix} U_{xx} & U_{xy} \\ U_{yx} & U_{yy} \end{bmatrix} = \begin{bmatrix} 12 & 5 \\ 5 & 18 \end{bmatrix}$

(b) $z(x,y) = 3(11x - 6y)^2$

Ans. $H = \begin{bmatrix} z_{xx} & z_{xy} \\ z_{yx} & z_{yy} \end{bmatrix} = \begin{bmatrix} 726 & -396 \\ -396 & 216 \end{bmatrix}$

3. Find the total differentials for the following functions:

(a) $z = 4x^3 - 13xy - 6y^5$

Ans. $dz = (12x^2 - 13y)dx + (-13x - 30y^4)dy$

(b) $z = (2x^2 - y)(3x - 4y^3)$

Ans. $dz = (18x^2 - 16xy^3 - 3y)dx + (-24x^2y^2 + 16y^3 - 3x)dy$

(c) $z = \frac{7y^3}{(5x-2y)}$

Ans. $dz = \frac{(-35y^3)dx+(105xy^2-28y^3)dy}{(5x-2y)^2}$

Let $u \equiv 7y^3$ and $v \equiv 5x - 2y$

$\Rightarrow du = 21y^2dy$; $dv = 5dx - 2dy$.

$$dz = \frac{vdu - udv}{v^2} = \frac{(5x-2y)(21y^2dy) - (7y^3)[5dx - 2dy]}{(5x-2y)^2}$$

$$= \frac{[105xy^2 - 42y^3 + 14y^3]dy - 35y^3dx}{(5x-2y)^2}$$

$$dz = \frac{(-35y^3)dx + (105xy^2 - 28y^3)dy}{(5x-2y)^2}$$

(d) $z = 8x^{\frac{1}{2}}y^{\frac{1}{4}}w^{\frac{1}{4}}$

Ans.

$$dz = z_x dx + z_y dy + z_w dw$$

$$dz = (4x^{-1/2}y^{1/4}w^{1/4})dx + (2x^{1/2}y^{-3/4}w^{1/4})dy + (2x^{1/2}y^{1/4}w^{-3/4})dw.$$

4. Given the equation for the production isoquant

$$18[0.2K^{-0.4} + 0.8L^{-0.4}]^{-2.5} = 1936$$

Use the implicit function rule to find the marginal rate of technical substitution (MRTS) of L for K.

Ans.

Let $F(K,L) \equiv 18[0.2K^{-0.4} + 0.8L^{-0.4}]^{-2.5} - 1936 = 0.$

Using implicit function rule, $\frac{dK}{dL} = -\frac{F_L}{F_K}.$

$$F_L = (18)(-2.5)[0.2K^{-0.4} + 0.8L^{-0.4}]^{-3.5}(-0.32)L^{-1.4}$$
$$= 14.4[0.2K^{-0.4} + 0.8L^{-0.4}]^{-3.5} \times L^{-1.4}$$

$$F_K = (18)(-2.5)[0.2K^{-0.4} + 0.8L^{-0.4}]^{-3.5}(-0.08) \cdot K^{-1.4}$$
$$= 3.6[0.2K^{-0.4} + 0.8L^{-0.4}]^{-3.5} \times K^{-1.4}$$

$$\Rightarrow \underbrace{\frac{dK}{dL}}_{MRTS} = -\frac{F_L}{F_K} = -\frac{14.4\,\cancel{[0.2K^{-0.4} + 0.8L^{-0.4}]^{-3.5}} \cdot L^{-1.4}}{3.6\,\cancel{[0.2K^{-0.4} + 0.8L^{-0.4}]^{-3.5}} \cdot K^{-1.4}} = -4\left(\frac{K}{L}\right)^{1.4}$$

5. Let the demand for a commodity be

$Q_d = D(P, Y_0, t_0),$ $\qquad \frac{\partial D}{\partial P} < 0; \frac{\partial D}{\partial Y_0} > 0; \frac{\partial D}{\partial t_0} > 0$

$Q_s = S(P, T_0),$ $\qquad \frac{\partial S}{\partial P} > 0; \frac{\partial S}{\partial T_0} < 0$

Where P is the price, Y_0 is the consumer's income, t_0 is the taste for the commodity and T_0 is the tax on the commodity. All derivatives are continuous. Find $\frac{\partial P^*}{\partial t_0}, \frac{\partial P^*}{\partial T_0}, \frac{\partial Q^*}{\partial Y_0}$, and $\frac{\partial Q^*}{\partial T_0}$. Discuss their economic implications.

Ans.

Let P^* be the equilibrium price.

$$P^* = P^*(Y_0, t_0, T_0).$$

At Eqm, $D(P^*, Y_0, t_0) = S(P^*, T_0)$.

$$\Rightarrow F \equiv D(P^*, Y_0, t_0) - S(P^*, T_0) = 0. \quad : \text{Implicit function.}$$

- $$\frac{\partial P^*}{\partial t_0} = -\frac{F_{t_0}}{F_{P^*}} = -\frac{\overset{\oplus}{\partial D/\partial t_0}}{\underbrace{\underset{\ominus}{\partial D/\partial P^*} - \underset{\oplus}{\partial S/\partial P^*}}_{\ominus}} > 0.$$

$\Rightarrow$ As the taste increases, the equilibrium price rises.

- $$\frac{\partial P^*}{\partial T_0} = -\frac{F_{T_0}}{F_{P^*}} = -\frac{\overset{\oplus}{\boxed{-\overset{\ominus}{\partial S/\partial T_0}}}}{\underbrace{\frac{\partial D}{\partial P^*} - \frac{\partial S}{\partial P^*}}_{\ominus}} > 0$$

$\Rightarrow$ As the tax imposed on producer increases, the equilibrium price increases.

- $$\frac{\partial Q^*}{\partial Y_0} = \underset{\oplus}{\frac{\partial S}{\partial P^*}} \cdot \underset{\oplus}{\frac{\partial P^*}{\partial Y_0}} \quad [\because Q^* = S(P^*, T_0) = D(P^*, Y_0, t_0) \text{ and } P^* = P^*(Y_0, t_0, T_0)].$$

$$\frac{\partial P^*}{\partial Y_0} = -\frac{F_{Y_0}}{F_{P^*}} = -\frac{\overset{\ominus}{\partial D/\partial Y_0}}{\underset{\ominus}{\frac{\partial D}{\partial P^*}} - \underset{\oplus}{\frac{\partial S}{\partial P^*}}} > 0.$$

$$\therefore \frac{\partial Q^*}{\partial Y_0} = \frac{\partial S}{\partial P^*} \cdot \frac{\partial P^*}{\partial Y_0} > 0.$$

$\Rightarrow$ As income increases, the equilibrium quantity increases.

Double A

Questions 6-14 are from Sydsaeter and Hammond, 2008. Questions 15-16 are from Wainwright.
** indicates more difficult problems.*

6. The demand for money, *M*, in the United States for the period 1929-1952 has been estimated as

$$M = 0.14Y + 76.03(r-2)^{-0.84}, \quad (r > 2)$$

where Y is the annual national income, and r is the interest rate measured in percent per year. Find $\partial M/\partial Y$ and $\partial M/\partial r$ and discuss their signs.

Ans. $\partial M/\partial Y = 0.14 > 0$: Money demand increases with income.

$\partial M/\partial r = -0.84 \cdot 76.03(r-2)^{-1.84} = -63.87(r-2)^{-1.84} < 0$: Money demand decreases as the interest rate increases.

7. The demand for a product depends on the price *p* of the product and on the price *q* charged by a competing producer

$$D(p,q) = a - bpq^{-\alpha}$$

where *a*, *b*, and α are positive constants with α < 1. Find $D_p'(p,q)$ and $D_q'(p,q)$, and comment on the signs of the partial derivatives.

Ans. $D_p'(p,q) = bq^{-\alpha} < 0$, showing that the demand decreases as its own price increases.

$D_q'(p,q) = -b\alpha pq^{-\alpha-1} > 0$, showing that the demand increases as the price of a competing product increases.

8. Let *x* and *y* be the populations of two cities and *d* the distance between them. Suppose that the number of travelers *T* between the cities is given by

$$T = kxy/d^n \qquad \text{(k and n are positive constants.)}$$

Find $\partial T/\partial x$, $\partial T/\partial y$, and $\partial T/\partial d$, and discuss their signs.

Ans. $\partial T/\partial x = \frac{ky}{d^n} > 0$ and $\partial T/\partial y = \frac{kx}{d^n} > 0$, suggesting that the number of travelers increases as the size of either city increases.

$\partial T/\partial d = -\frac{nkxy}{d^{n+1}} < 0$, suggesting that the number of travelers decreases as the distance between the two cities increases.

*9. Let *D(p, m)* indicate a typical consumer's demand for a particular commodity, as a function of its price *p* and the consumer's own income *m*. Show that the proportion *pD/m* of income spent on the commodity increases with income if $\varepsilon_m > 1$ (in which case the good is a "luxury" whereas it is a "necessity" if $\varepsilon_m < 1$).

Ans. $\frac{\partial}{\partial m}\left(\frac{pD(p,m)}{m}\right) = p\frac{mD_m - D}{m^2} = \frac{pD}{m^2}\left[m\frac{\partial D}{\partial m}\cdot\frac{1}{D} - 1\right] = \frac{pD}{m^2}[\varepsilon_m - 1] > 0\ iff\ \varepsilon_m > 1.$

So, *pD/m* increases with *m* if $\varepsilon_m > 1$ (“luxury” goods).

Similarly, one can show that $\frac{\partial}{\partial m}\left(\frac{pD(p,m)}{m}\right) = \frac{pD}{m^2}[\varepsilon_m - 1] < 0\ iff\ \varepsilon_m < 1$ (“necessity” goods).

10. The annual herring catch is given by the function $Y(K,S) = 0.06157K^{1.356}S^{0.562}$ of the catching effort (K) and the herring stock (S).

(a) Find $\partial Y/\partial K$ and $\partial Y/\partial S$.

Ans. a) $\partial Y/\partial K \approx 0.083K^{0.356}L^{0.562}$ and $\partial Y/\partial S \approx 0.035K^{1.356}L^{-0.438}$.

(b) If K and S are both doubled, what happens to the catch?

Ans. If K and S are both doubled, the catch becomes $2^{1.356+0.562} = 2^{1.918} \approx 3.779$ times higher.

11. Find the marginal rate of substitution (MRS) between *y* and *x* when:

(a) $U(x,y) = 2x^{0.4}y^{0.6}$ Ans. $MRS_{yx} = \frac{U_x}{U_y} = \frac{2y}{3x}$

(b) $U(x,y) = xy + y$ Ans. $MRS_{yx} = \frac{U_x}{U_y} = \frac{y}{x+1}$

(c) $U(x,y) = 10(x^{-2} + y^{-2})^{-4}$ Ans. $MRS_{yx} = \frac{U_x}{U_y} = \left(\frac{y}{x}\right)^3$

12. Suppose that a firm produces $Q = f(L) = \sqrt{L}$ units of commodity using L units of labor. Assume that *f'(L) > 0* and *f''(L) < 0*, so f is strictly increasing and strictly concave. If the firm gets *P* baht per unit produced and pays *w* baht for a unit of labor, write down the profit function, and find the first-order condition for profit maximization at *L* > 0.*

Ans. Profit function: $(L) = Pf(L) - wL$.

F.O.C. $Pf'(L) - w = 0 \Rightarrow \frac{P}{2\sqrt{L^*}} = w \Rightarrow L^* = \frac{P^2}{4w^2}$.

*13. Suppose production X depends on the number of workers N according the formula

$$X = Ng(\frac{\varphi(N)}{N})$$

where g and φ are given differentiable functions. Find expressions for dX/dN and d^2X/dN^2.

Ans. $\frac{dX}{dN} = g(u) + Ng'(u)\frac{du}{dN} = g(u) + g'(u)(\varphi'(N) - u)$, where $u = \frac{\varphi(N)}{N}$

$$\frac{d^2X}{dN^2} = g'(u)\frac{du}{dN} + g''(u)\frac{du}{dN}(\varphi'(N) - u) + g'(u)\left(\varphi''(N) - \frac{du}{dN}\right)$$

$$= \frac{1}{n}g''\left(\frac{\varphi(N)}{N}\right)\left[\varphi'(N) - \frac{\varphi(N)}{N}\right]^2 + g'\left(\frac{\varphi(N)}{N}\right)\varphi''(N).$$

*14. An equilibrium model of labor demand and output pricing leads to the following system of equations:

$$pF'(L) - w = 0$$

$$pF(L) - wL - B = 0 \qquad (*)$$

Suppose that F is twice differentiable with $F'(L) > 0$ and $F''(L) < 0$, and all the variables are positive. Treat w and B as exogenous, so that p and L are endogenous variables which are functions of w and B.

(a) Find expressions for $\partial p/\partial w$, $\partial p/\partial B$, $\partial L/\partial w$, and $\partial L/\partial B$ by implicit differentiation.

Ans. $\partial p/\partial w = \frac{L}{F(L)}$; $\qquad \partial p/\partial B = \frac{1}{F(L)}$;

$\partial L/\partial w = \frac{F(L) - LF'(L)}{pF(L)F''(L)}$; $\qquad \partial L/\partial B = -\frac{F'(L)}{pF(L)F''(L)}$

Derivations of the above expressions:

$$p.F'(L) - w = 0 \qquad — ①$$

$$pF(L) - wL - B = 0 \qquad — ②$$

$$① \Rightarrow \quad p.F''(L).dL + F'(L)\,dp = dw \qquad — ③$$

$$② \Rightarrow \quad \underbrace{p.F'(L)}_{=w}dL + F(L)\,dp - w\,dL - L\,dw = dB$$

$$\cancel{w\,dL} + F(L)\,dp - \cancel{w\,dL} - L\,dw = dB$$

$$F(L)\,dp - L\,dw = dB$$

$$\Rightarrow dp = \frac{dB + L\,dw}{F(L)} \quad \text{—④}$$

Note: $\boxed{L = L(w, B)}$

Sub dp in ①:

$$p \cdot F''(L)\,dL + F'(L) \cdot \frac{[dB + L\,dw]}{F(L)} = dw$$

$$pF''(L) \cdot F(L)\,dL + F'(L)\,dB + L \cdot F'(L)\,dw = F(L)\,dw$$

$$pF''(L) \cdot F(L) \cdot dL = [-LF'(L) + F(L)]\,dw - F'(L)\,dB$$

$$dL = \underbrace{\frac{[-LF'(L) + F(L)]}{p \cdot F''(L)F(L)}}_{= L_w = \frac{\partial L}{\partial W}} dw \underbrace{- \frac{F'(L)}{p \cdot F''(L) \cdot F(L)}}_{= L_B = \frac{\partial L}{\partial B}} \cdot dB$$

$$\boxed{\text{Recall: } dL = L_w \cdot dw + L_B \cdot dB}$$

Thus, $\boxed{\frac{\partial L}{\partial W} = \frac{F(L) - LF'(L)}{p \cdot F''(L) \cdot F(L)}}$; $\boxed{\frac{\partial L}{\partial B} = \frac{-F'(L)}{p \cdot F''(L) \cdot F(L)}}$

To find $\frac{\partial p}{\partial w}$ and $\frac{\partial p}{\partial B}$, recall that $P = P(w, B)$. So,

$$dp = \frac{\partial p}{\partial w} \cdot dw + \frac{\partial p}{\partial B} \cdot dB.$$

From ④, we have $dp = \underbrace{\frac{L}{F(L)}}_{P_w} dw + \underbrace{\frac{1}{F(L)}}_{P_B} dB.$

Thus, $\boxed{\frac{\partial p}{\partial w} = \frac{L}{F(L)}}$ and $\boxed{\frac{\partial p}{\partial B} = \frac{1}{F(L)}}$.

(b) What can be said about the signs of these partial derivatives? Show that $\partial L/\partial w < 0$.

Ans. Since $p > 0$, $F'(L) > 0$, and $F''(L) < 0$, $\partial p/\partial w > 0$; $\partial p/\partial B > 0$; $\partial L/\partial B > 0$.

We can show that $F(L) - LF'(L) = B/p > 0$. Thus, $\partial L/\partial w < 0$.

15. If $z = x^3y^2 + x^2y^4 - 3xy$ and $x = r + 3s$ and $y = 2r - s$, then determine $\frac{\partial z}{\partial s}$ when r = 1 and s = 0.

Ans.

$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s} = (3x^2y^2 + 2xy^4 - 3y)(3) + (2x^3y + 4x^2y^3 - 3x)(-1)$

At r = 1 and s = 0, x = 1, y =2. $\frac{\partial z}{\partial s} = 126 - 31 = 95$

16. If $x^2 + xy + yz + x^2 = 6$, then:

(a) Find $\frac{\partial z}{\partial y}$

Ans. $\frac{\partial z}{\partial y} = -\frac{x+z}{y}$

(b) Evaluate $\frac{\partial z}{\partial y}$ at x =1, y=2, z =1.

Ans. $\frac{\partial z}{\partial y} = -\frac{x+z}{y} = -\frac{2}{2} = -1$