

Macroeconomics

Lecture 3

Bellman's Equations

- Define the (T+1)-period value function

$$W_{T+1}(x_0) = \max_{c_0, \dots, c_T} [U_0(x_0, c_0) + U_1(x_1, c_1) + \dots + U_T(x_T, c_T)] \quad (1)$$

Subject to $x_{t+1} = g_t(x_t, c_t), \quad t = 0, 1, \dots, T; \quad x_0 \text{ given.}$

Problem (1) is the same as

$$W_{T+1}(x_0) = \max_{c_0} \left[U_0(x_0, c_0) + \max_{c_1} \left[U_1(x_1, c_1) + \dots \max_{c_T} [U_T(x_T, c_T)] \right] \right] \quad (2)$$

Subject to $x_{t+1} = g_t(x_t, c_t), \quad x_0 \text{ given.}$

- (1) Solving for $c_T = h_T(x_T)$ to optimize $W_1(x_T)$,
- (2) Solving for $c_{T-1} = h_{T-1}(x_{T-1})$ to optimize $W_2(x_{T-1})$,
Continue to repeat this process until $t=0$.

Self-enforcing property

- The optimal policies $c_t = h_t(x_t)$, $t=0,1,2,\dots,T$, from problem (1) will have the same values of

$$c_s = h_s(x_s), \quad t=s, s+1, s+2, \dots, T,$$

as those optimal policies obtained from the following problem: for $s > 0$

$$\max_{c_s, \dots, c_T} [U_s(x_s, c_s) + U_{s+1}(x_{s+1}, c_{s+1}) + \dots + U_T(x_T, c_T)] \quad (3)$$

$$\text{Subject to} \quad x_{t+1} = g_t(x_t, c_t), \quad t = s, s+1, \dots; \quad x_s \text{ given.}$$

- Then, these optimal policies are said to be “time consistent”.

Policy function that does not vary over time

- Assume that: $U_t(x_t, c_t) = \beta^t \cdot U(x_t, c_t), \quad 0 < \beta < 1$
- $g_t(x_t, c_t) = g(x_t, c_t), \quad (4)$
- Bellman's equation becomes (for $j=1, 2, \dots, T$)

$$\begin{aligned}
 W_{j+1}(x_{T-j}) &= \max_{c_{T-j}} \left[\beta^{T-j} U(x_{T-j}, c_{T-j}) + W_j(x_{T-j+1}) \right] \\
 \beta^{j-T} W_{j+1}(x_{T-j}) &= \max_{c_{T-j}} \left[U(x_{T-j}, c_{T-j}) + \beta \cdot \beta^{j-T-1} W_j(x_{T-j+1}) \right] \\
 V_{j+1}(x_{T-j}) &= \max_{c_{T-j}} \left[U(x_{T-j}, c_{T-j}) + \beta \cdot V_j(x_{T-j+1}) \right] \quad (5)
 \end{aligned}$$

- where

$$V_{j+1}(x_{T-j}) = \beta^{j-T} W_{j+1}(x_{T-j})$$

$$\text{If } j=T, \text{ then } V_{T+1}(x_0) = W_{T+1}(x_0)$$

Policy function that does not vary over time

The Bellman's equation becomes, (for $j=1,2,\dots,T$)

$$V_{j+1}(x_{T-j}) = \max_{c_{T-j}} \left[U(x_{T-j}, c_{T-j}) + \beta V_j(x_{T-j+1}) \right] \quad (6)$$

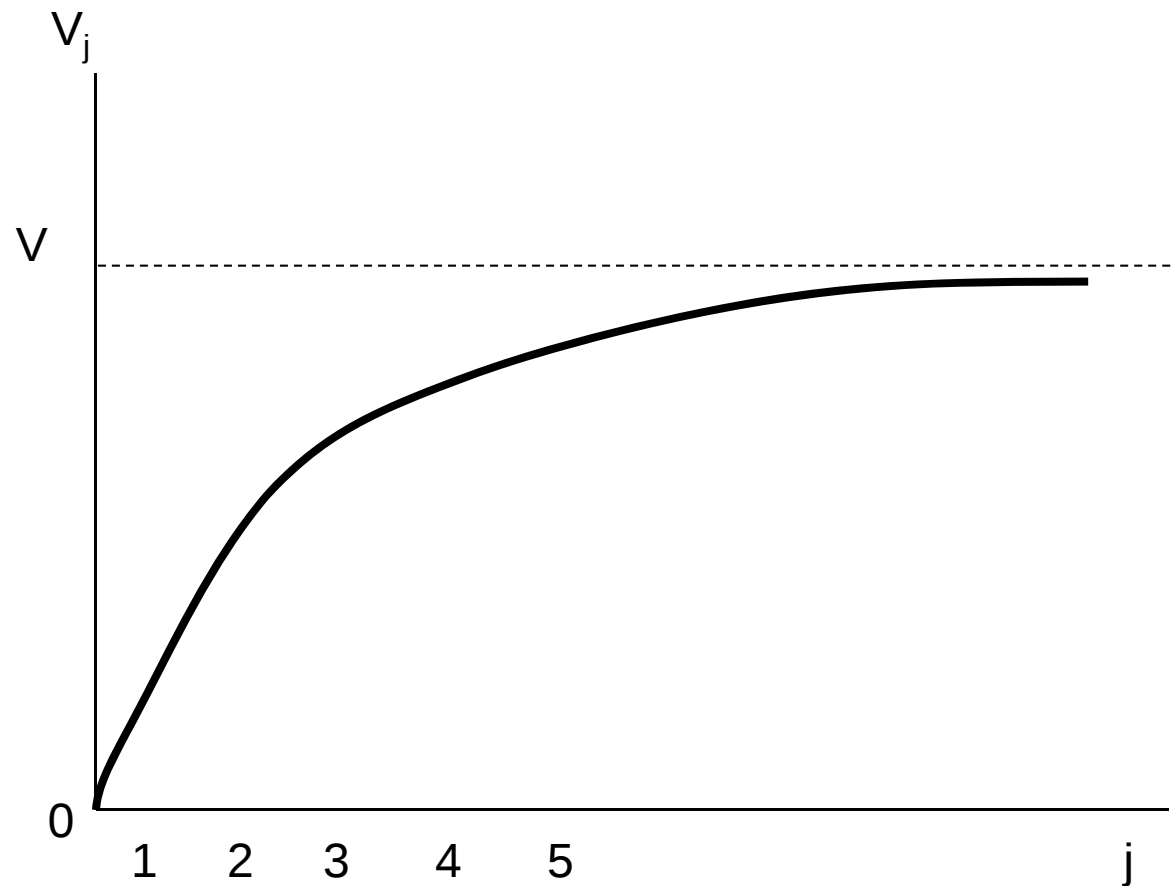
Subject to $x_{T-j+1} = g(x_{T-j}, c_{T-j})$, x_{T-j} given

- In the case that $V = \lim_{j \rightarrow \infty} V_j$, then

$$V(x) = \max_c \left[U(x, c) + \beta V(\tilde{x}) \right] \quad (7)$$

Subject to $\tilde{x} = g(x, c)$, x given.

Policy function that does not vary over time



Discounted Dynamic Programming Problem

- The limiting value function V that solves (7) is the optimal value function for the infinite horizon problem

$$\max_{\{c_t\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t U(x_t, c_t) \quad (8)$$

Subject to $x_{t+1} = g(x_t, c_t), \quad x_0 \text{ given}$

- There is a unique and time-invariant optimal policy
- $c_t = h(x_t), t=0,1,2,\dots,T$
- Where $h(\cdot)$ is chosen to maximize the right hand side of (7), and

Discounted Dynamic Programming Problem

the limiting value function V is differentiable with respect to x . (This is the same as the marginal condition (1.4d) in Lecture #1). Hence,

$$V'(x) = \frac{\partial U[x, h(x)]}{\partial x} + \frac{\partial U[x, h(x)]}{\partial h(x)} \cdot \frac{\partial h(x)}{\partial x} + \beta \frac{\partial g[x, h(x)]}{\partial x} V'(g[x, h(x)]) \quad (9)$$