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EE431/438 Economics of Financial Markets and Institutions

Exercise 4 : Arbitrage Pricing Theory

Please submit at the BE office, 5th floor department of Economics building.

Late submission will not be accepted.

1. Assuming that the market is complete and perfect, what is the arbitrage equilibrium price of asset D, E, G, H in the example below? Hint: At equilibrium, no arbitrage opportunity exists.

	State 1	State 2	Price
Asset A	1	0	0.4
Asset B	0	1	0.5
Asset C	2	0	$=$ (..2..$\times$..0.4..) + (..0..$\times$..0..) =..0.8..
Asset D	0	2	$=$ (..0..$\times$..0.4..) + (..2..$\times$..0.5..) =..1..
Asset E	2	2	$=$ (..2..$\times$..0.4..) + (..2..$\times$..0.5..) =..1.8..
Asset G	3	1	$=$ (..3..$\times$..0.4..) + (..1..$\times$..0.5..) =..1.7..
Asset H	1	1	$=$ (..1..$\times$..0.4..) + (..1..$\times$..0.5..) =..0.9..

Read the following note.

- Notice that, at equilibrium, price of an asset $= p_1Q_1 + p_2Q_2$, where
- p_1 is the price of security paying \$1 if state 1 occurs and nothing if any other state occurs (the price of security A in this example),
  - p_2 is the price of security paying \$1 if state 2 occurs and nothing if any other state occurs (the price of security B in this example),
  - Q_1 is the quantity of dollars the asset paid if state 1 occurs,
  - Q_2 is the quantity of dollars the asset paid if state 2 occurs.
- If there are n states, price of an asset $=\sum_{i=1}^{n} p_iQ_i$, where
  - p_i is the price of security paying \$1 if state i occurs and nothing if any other state occurs,
  - Q_i is the quantity of dollars the asset paid if state i occurs.
- Security that pays \$1 if a specified state occurs and nothing if any other state occurs is called "a pure security". In this example, security A and security B are pure securities.
- It is not necessary that $p_1 + p_2 = 1$. There are more than 2 states in this example. The given information is sufficient to calculate the price of any asset which pays some money when state 1 and/or state 2 occur(s).

2. Using the information given in question 1, is there any arbitrage opportunity if the price of security C is equal to 1? If yes, explain how an investor can make an arbitrage profit.

   Answer.

   Security C pays 2 units if state 1 occurs and nothing otherwise. Security C and a portfolio consisting of 2 of security A has the same pattern of payoffs. Both pay 2 units if state 1 occurs and nothing otherwise. Therefore, they should be priced identically (according to the law of one price or the single-price law of market). Otherwise, everyone would want to buy the security or the portfolio with the lower price and to sell the security or the portfolio with the higher price. The price of security C is equal to 1. The price of security C is *greater* than the price of a portfolio consisting of 2 of security A ($1 > (2 \times 0.8), 1 > 0.8$). We should *short-sell* the more expensive one, *security C*, at price 1 and *buy* the less expensive one, *the portfolio consisting of 2 of security A*, at price 0.8. We would realize *a positive net cash flow of 1 - 0.8 = 0.2 units. At the end of the period, we could at no risk.* We can take our payoff from owning the portfolio consisting of 2 of security A (2 units if state 1 occurs and nothing otherwise) to *exactly* repay our obligation from selling one security C (2 units if state 1 occurs and nothing otherwise). *The positive net cashflow at the beginning of the period represents a riskless arbitrage profit opportunity.* This arbitrage opportunity is inconsistent with market equilibrium.

3. Using the information given in question 1, is there any arbitrage opportunity if the price of security C is equal to 0.6? If yes, how an investor can make an arbitrage profit? Explain.

   Answer.

   Security C pays 2 units if state 1 occurs and nothing otherwise. Security C and a portfolio consisting of 2 of security A has the same pattern of payoffs. Both pay 2 units if state 1 occurs and nothing otherwise. Therefore, they should be priced identically (according to the law of one price or the single-price law of market). Otherwise, everyone would want to buy the security or the portfolio with the lower price and to sell the security or the portfolio with the higher price. The price of security C is equal to 0.6. The price of security C is *lower* than the price of a portfolio consisting of 2 of security A ($0.6 < (2 \times 0.8), 0.6 < 0.8$). We should *short-sell* the more expensive one, *the portfolio consisting of 2 of security A*, at price 0.8 and *buy* the less expensive one, *security C*, at price 0.6. We would realize *a positive net cash flow of 1 - 0.8 = 0.2 units. At the end of the period, we could at no risk.* We can take our payoff from owning the portfolio consisting of 2 of security A (2 units if state 1 occurs and nothing otherwise) to *exactly* repay our obligation from selling one security C (2 units if state 1 occurs and nothing otherwise). *The positive net cashflow at the beginning of the period represents a riskless arbitrage profit opportunity.* This arbitrage opportunity is inconsistent with market equilibrium.

4. You are given the following information.

Security	Payoff		Security Prices
	State 1	State 2	
j	\$0	\$8	4
k	9	4	5
Asset 1	1	0	$\frac{1}{3}$
Asset 2	0	1	$\frac{1}{2}$
Asset C	5	4	$3.\dot{6}$ =3.67

   (a) What are the prices of asset 1 and asset 2? Hint: See the note in question 1.

   Answer.

   Note Asset 1 is a pure security for state 1 (the security that pays \$1 if state 1 occurs and nothing if any other state occurs). Asset 2 is a pure security for state 2 (the security that pays \$1 if state 2 occurs and nothing if any other state occurs).

Let p_1 be the price of Asset 1 and p_2 be the price of Asset 2. At equilibrium, there is no arbitrage opportunity. Use price of an asset $= p_1 Q_1 + p_2 Q_2$ (See the note in question 1), get

$4 = (0 \times p_1) + (8 \times p_2),$

$5 = (9 \times p_1) + (4 \times p_2).$

Solving this system of two equations for p_1 and p_2 and get $p_1 = \frac{1}{3}$ and $p_2 = \frac{1}{2}$.

(b) What is the equilibrium price of a third security C? Hint: See the note in question 1.

Answer.

Let p_C be the price of asset C. At equilibrium, there is no arbitrage opportunity

Use price of an asset $= p_1 Q_1 + p_2 Q_2$ (See the note in question 1) and $p_1 = \frac{1}{3}$ and $p_2 = \frac{1}{2}$, get

$p_C = (5 \times \frac{1}{3}) + (4 \times \frac{1}{2}) = 1.67 + 2 = 3.67.$

5. The returns on stocks A and B are determined by the following one-factor model:

$ER_A = 0.05 + 0.5F_1$; asset A has $\beta_{A1} = 0.5$.

$ER_B = 0.02 - 0.25F_1$; asset B has $\beta_{B1} = -0.25$.

Notation: β_{j1} and β_{j1} for j = A; B denote the responses of the rates of return on assets A and B to the factor 1 (F_1).

(a) Determine the portfolio weights you need to place on A and B in order to construct a portfolio which has zero exposure to the factor 1 (F_1) (a risk-free portfolio). What is the expected return of the risk-free portfolio?

Answer.

Let w_A be the weight of asset A and w_B be the weight of asset B in a portfolio. Then, expected return of the portfolio can be written as follows.

$$\begin{aligned} E(R_p) &= aE(X) + bE(Y), \\ E(R_p) &= w_A E(R_A) + w_B E(R_B), \\ &= w_A(0.05 + 0.5F_1) + w_B(0.02 - 0.25F_1), \\ &= (0.05w_A + 0.02w_B) + (0.5w_A - 0.25w_B)F_1. \end{aligned}$$

A riskless portfolio must have beta equal to zero. Therefore,

$0.5w_A - 0.25w_B = 0 \Rightarrow$ equation (1).

The sum of the weights must always be 1.

$w_A + w_B = 1 \Rightarrow$ equation (2),

Solving the system of equations (equation (1) and equation (2)) for w_A and w_B and get $w_A = \frac{1}{3}$ and $w_B = \frac{2}{3}$.

Substitute $w_A = \frac{1}{3}$ and $w_B = \frac{2}{3}$ in to $E(R_p) = (0.05w_A + 0.02w_B) + (0.5w_A - 0.25w_B)F_1$ and get $E(R_p) = \frac{0.05 + 2(0.04)}{3} = 0.03$.

Hence, to construct a riskless portfolio, the weight placed on asset A must be equal to $\frac{1}{3}$ and the weight placed on asset B must be equal to $\frac{2}{3}$. The return of the risk-free portfolio is then equal to 0.03 or 3%.

(b) From (a), is there any abritrage opportunity exist if the market risk-free interest rate is 1%? If yes, explain how an investor can make an arbitrage profit.

Answer.

The risk-free portfolio in (a) and the risk-free asset avaiable in the market (which pays the market risk-free interest rate) are both riskless. Their returns are certain, regardless of the value of F_1. They have the same value of beta. They should yield the same rate of return. Otherwise, everyone would want to buy the security or the portfolio with the lower price (higher yields) and to sell the security or the portfolio with the higher price (lower yields). The rate of return of the risk-free asset porfolio in (a) is equal to 0.03. The rate of return of the risk-free asset porfolio in (a) is *higher* than tje market risk-free interest rate (for example, the interest rate on treasury bill). $3\% > 1\%$. $0.03 > 0.01$. We should *short-sell* the one that *pays the lower rate of retur*n, the risk-free asset available in the market. In other words, we borrow from the market and pay the market risk-free interest rate. After that, we use the proceeds to *buy* the one that *pays the higher rate of return, risk-free portfolio in (a).* We get 3% rate of return from investing in a risk-free portfolio in (a). We would realize a profit of $(3\% - 1\%) = 2\%$. This arbitrage opportunity is inconsistent with market equilibrium.

(c) Determine the portfolio weights you need to place on A and B in order to construct a portfolio which has a unit exposure to the factor 1 (F_1). What is the expected return of the portfolio which has a unit exposure to the factor 1?

Answer.

Let w_A be the weight of asset A and w_B be the weight of asset B in a portfolio. Then, expected return of the portfolio can be written as follows.

$$\begin{aligned} E(R_p) &= aE(X) + bE(Y), \\ E(R_p) &= w_A E(R_A) + w_B E(R_B), \\ &= w_A(0.05 + 0.5F_1) + w_B(0.02 - 0.25F_1), \\ &= (0.05w_A + 0.02w_B) + (0.5w_A - 0.25w_B)F_1. \end{aligned}$$

Given that the portfolio must have beta equal to one. Therefore,

$0.5w_A - 0.25w_B = 1 \Rightarrow$ equation (1).

The sum of the weights must always be 1.

$w_A + w_B = 1 \Rightarrow$ equation (2),

Solving the system of equations (equation (1) and equation (2)) for w_A and w_B and get $w_A = \frac{5}{3}$ and $w_B = -\frac{2}{3}$.

Substitute $w_A = \frac{5}{3}$ and $w_B = -\frac{2}{3}$ in to $E(R_p) = (0.05w_A + 0.02w_B) + (0.5w_A - 0.25w_B)F_1$ and get $E(R_p) = \left(\frac{0.05(5) + (0.02)(-2)}{3}\right) + \left(\frac{0.5(5) - 0.25(-2)}{3}\right) F_1 = 0.07 + F_1$.

Hence, to construct a portfolio which has a unit exposure to the factor 1 (F_1), the weight placed on asset A must be equal to $\frac{5}{3}$ and the weight placed on asset B must be equal to $-\frac{2}{3}$ (we have to short sell asset B.). The expected return of the portfolio which has a unit exposure to the factor 1 (F_1) is then equal to $0.07 + F_1$.

(d) Derive an equation for the equilibrium expected rate of return of an asset K which has $\beta_{K1} = b_K$. (Hint: $ER_i = R_f + (\lambda_1 - R_f)\beta_{i1}$, where λ_1 is the expected rate of return of the portfolio which as a unit exposure to the factor 1.)

Answer.

Making use $ER_i = R_f + (\lambda_1 - R_f)\beta_{i1}$, $R_f = 0.03$ (from question (a) and $E(R_p) = \lambda_1 = 0.07 + F_1$ (from question (c) , and get

$$\begin{aligned} E(R_i) &= 0.03 + ((0.07 + F_1) - 0.03)\,\beta_{i1}, \\ &= 0.03 + (0.04 + F_1)\,\beta_{i1}, \\ &= (0.03 + 0.04\beta_{i1}) + \beta_{i1}F_1. \end{aligned}$$

The equilibrium expected rate of return of the asset K is equal to $(0.03 + 0.04b_K) + b_K F_1$.

6. Suppose $R_i = 0.05 + 0.5F_1 = 0.075$. Find the value of F_1.

   Answer.

   Simply solve the equation $R_i = 0.05 + 0.5F_1 = 0.075$ for F_1 . Then, get

   $0.5F_1 = 0.075 - 0.05,\ F_1 = \frac{0.025}{0.5} = 0.05.$

7. The following information is provided for a stock market in which asset returns respond to one factor:

Asset i	β_{i1}	Expected Return on Asset i ($E(R_i)$)
Asset M	1	$0.10 = E(R_m) = \lambda_1$
Asset R_f	0	$0.06 = R_f$
Asset A	0.2	?

   Find the expected rate of return on asset A. (Hint: $ER_i = R_f + (\lambda_1 - R_f)\beta_{i1}$, where λ_1 is the expected rate of return of the portfolio which as a unit exposure to the factor 1 ($\beta_{i1} = 1$).)

   Answer.

   Use $ER_i = R_f + (\lambda_1 - R_f)\beta_{i1} = R_f + (E(R_m) - R_f)\,\beta_{i1}$, get

   $ER_A = 0.06 + (0.10 - 0.06)0.2 = 0.06 + (0.04)0.2) = 0.068 = 6.8\%.$

   Note that $(\lambda_1 - R_f)$ is the risk premium for factor 1. The risk premium for factor 1 is equal to $0.10 - 0.06 = 0.04$.