Relations & Functions II

TU152: Fundamental Mathematics

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Definition (Function)

A function f from a set X to a set Y, denoted $f : X \to Y$, is a relation from X to Y that satisfies the following two properties.

(1) Every element in X is related to some element in Y. I.e.

$$\forall x \in X, \exists y \in Y, \text{ such that } (x, y) \in f.$$

(2) No element in X is related to more than one element in Y. I.e.

$$\forall x \in X, \text{ if } (x, y_1), (x, y_2) \in f \text{ where } y_1, y_2 \in Y, \text{ then } y_1 = y_2$$

The set X is called the **domain** of f and Y is called the **co-domain**of f.

- The set of all values of f taken together is called the **range** of f or the **image** of X under f. I.e.,

  $$\text{range of } f = \text{image of } X \text{ under } f = \{y \in Y | y = f(x), \text{ forsome} x \in X\}.$$

- Given an element y in Y , there may exist elements in X with y as their image. If $f(x) = y$, then x is called a **preimage** of y or an **inverse image** of y. The set of all inverse images of y is called the **inverse image** of y. I.e.,

  $$\text{the inverse image of } y = \{x \in X | f(x) = y\}$$

A “function” is not **well defined** if it fails to satisfy at least one of the requirements for being a function.

Function

Let f be a function from a set X to a set Y, denoted $f : X \to Y$. Let $(x, y) \in f$,

- We say that "f sends x to y" or " f maps x to y" and
- write $f : x \to y$ or $f(x) = y$.
- The unique element to which f sends x is denoted $f(x)$ and is called
  - f of x, or
  - the output of f for the input x, or
  - the value of f at x, or
  - the image of x under f.

Arrow Diagrams of Functions

Recall that if X and Y are finite sets, you can define a function f from X to Y by drawing an arrow diagram. You make a list of elements in X and a list of elements in Y, and draw an arrow from each element in X to the corresponding element in Y. The arrow diagram that defines a function must have the following two properties.

1. Every element of X has an arrow coming out of it.
2. No element of X has two arrows coming out of it that point to two different elements of Y.

Function

Equality of Functions

If $F : X \to Y$ and $G : X \to Y$ are functions, then $F = G$ if, and only if, $F(x) = G(x)$ for all $x \in X$.

Definition (Graph)

Let A and B be subsets of $\mathbb{R}$. A function $f : A \to B$ is called a real-valued function of a real variable. In this case, each ordered pair $(x, f(x))$ can be represented by a point in the Cartesian plane. The collection of all such points is called the **graph of** f.

Definition (Increasing/Decreasing Function)

Let X be the domain of a function f and $S \subset X$.

- f is increasing on S if and only if, for all $x_1, x_2 \in S$, if $x_1 < x_2$, then $f(x_1) < f(x_2)$.
- f is decreasing on S if and only if, for all $x_1, x_2 \in S$, if $x_1 < x_2$, then $f(x_1) > f(x_2)$.

Example:

(1) Show that the function $f : \mathbb{R} \to \mathbb{R}$, $f(x) = 2x - 3$ is increasing on $\mathbb{R}$.

(2) Show that the function $f : \mathbb{R} \to \mathbb{R}$, $f(x) = \frac{x+2}{x+1}$ is decreasing on $(-\infty, -1) \cup (-1, \infty)$.

Examples of Functions

- **The Identity Function on a Set:** Given a set X, define a function I_X from X to X by
$$I_X(x) = x$$
for all x in X.
- **Sequences:** The formal definition of sequences specifies that an infinite sequence is a function defined on the *set of integers that are greater than or equal to a particular integer*. E.g. the sequence denoted
$$1, -\frac{1}{2}, \frac{1}{3}, -\frac{1}{4}, \frac{1}{5}, \ldots, \frac{(-1)^n}{n+1}, \ldots.$$
can be thought of as the function f from the nonnegative integers to the real numbers that associates
$$0 \to 1, 1 \to -\frac{1}{2}, 2 \to \frac{1}{3}, 3 \to -\frac{1}{4}, 4 \to \frac{1}{5}, \ldots, n \to \frac{(-1)^n}{n+1}, \ldots.$$
That is, $f : \mathbb{Z} \cup \{0\} \to \mathbb{R}$ is the function defined as follows:
$$f(n) = \frac{(-1)^n}{n+1}.$$
Define a function g from the positive integers to the real numbers for the above sequence, $g : \mathbb{Z} \to \mathbb{R}$,
$$g(x) =$$

Examples of Functions

- **A Function Defined on a Power Set:** $P(A)$ denotes the set of all subsets of the set A. Define a function $F : P(a, b, c) \to \mathbb{Z} \cup \{0\}$ as follows. For each $X \in P(a, b, c)$,

$$F(X) = \text{the number of elements in } X.$$

  Draw an arrow diagram for F.
- **Functions Defined on a Cartesian Product:** E.g., define functions $M : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ and $R : \mathbb{R} \times \mathbb{R} \to \mathbb{R} \times \mathbb{R}$ as:
  for all ordered pairs (a, b) of real numbers, $M(a, b) = ab$ and $R(a, b) = (-a, b)$.
  Find the following:
  $M(-1, -1) =$
  $R(-1, -1) =$
  $M(-\sqrt{2}, \sqrt{2}) =$
  Exercise: Explain why each of M and R is indeed a function.

Injective (one-to-one) functions

Definition (Injective or one-to-one functions)

Let F be a function from a set X to a set Y. F is **one-to-one** or **injective** if, and only if, for all elements x_1 and x_2 in X,

$$\text{if } F(x_1) = F(x_2), \text{ then } x_1 = x_2,$$

or, equivalently (by contrapositive),

$$\text{if } x_1 \neq x_2, \text{ then } F(x_1) \neq F(x_2).$$

A function $F : X \to Y$ is **not** one-to-one if, and only if,

$$\exists x_1, x_2 \in X \text{ with } F(x_1) = F(x_2) \text{ and } x_1 \neq x_2.$$

In terms of arrow diagrams,

- A one-to-one function takes **distinct** points of the domain to **distinct** points of the co-domain.
- A function is not one-to-one will have at least two points of the domain taken to the same point of the co-domain.

Example: Let $X = \{1, 2, 3\}$ and $Y = \{a, b, c, d\}$. Define $H : X \to Y$ as follows: $H(1) = c$, $H(2) = a$, and $H(3) = d$. Define $K : X \to Y$ as follows: $K(1) = d$, $K(2) = b$, and $K(3) = d$. Determine if each of the functions H and K is one-to-one.

One-to-One Functions on Infinite Sets

Let f be a function defined on an infinite set X. By definition, f is one-to-one if, and only if, the following universal statement is true:

$$\forall x_1, x_2 \in X, \text{ if } f(x_1) = f(x_2) \text{ then } x_1 = x_2.$$

- To prove f is one-to-one, we will generally use the method of *direct proof*: suppose x_1 and x_2 are elements of X such that $f(x_1) = f(x_2)$ and show that $x_1 = x_2$.
- To show that f is **not** one-to-one, we will generally find elements x_1 and x_2 in X so that $f(x_1) = f(x_2)$ but $x_1 \neq x_2$.

Example: Show that the identity function I_X on a set X is injective.

Example: Define $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{Z} \to \mathbb{Z}$ by the rules
$f(x) = 4x - 1$ for all $x \in \mathbb{R}$, and $g(n) = n^2$ for all $n \in \mathbb{Z}$.

1. Is f one-to-one? Prove or give a counterexample.
2. Is g one-to-one? Prove or give a counterexample.

Example: Show that if $f : \mathbb{R} \to \mathbb{R}$ is increasing then f is one-to-one.

Function

Definition (Onto or Surjective Functions)

Let F be a function from a set X to a set Y. F is onto (or surjective) if, and only if, given any element y in Y, it is possible to find an element x in X with the property that $y = F(x)$. $F : X \to Y$ is onto if, and only if

$$\forall y \in Y, \exists x \in X \text{ such that } F(x) = y.$$

A function is not onto if, and only if,

$$\exists y \in Y \text{ such that } \forall x \in X, F(x) \neq y,$$

i.e., there is some element in Y that is not the image of any element in X.

Example:

Let $X = \{1, 2, 3, 4\}$ and $Y = \{a, b, c\}$.
Define $H : X \to Y$ as follows: $H(1) = c$, $H(2) = a$, $H(3) = c$, $H(4) = b$.
Define $K : X \to Y$ as follows: $K(1) = c$, $K(2) = b$, $K(3) = b$, and $K(4) = c$.
Determine whether each of the functions H and K are surjective (onto) or not.

Function

Example

1. Show that the function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = 3x - 5$ is surjective.
2. Show that the function $g : \mathbb{Z} \to \mathbb{Z}$ defined by $g(n) = 3n - 5$ is not surjective.

Example Let A and B be two nonempty sets.
The functions $P_A : A \times B \to A$ defined by $P_A(a, b) = a$ and $P_B : A \times B \to B$ defined by $P_B(a, b) = b$ are called projection functions. Show that P_A and P_B are surjective functions.

Function

Definition (one-to-one correspondence or bijection)

A **one-to-one correspondence** or **bijection** from a set X to a set Y is a function $F : X \to Y$ that is both *one-to-one* and *onto*.

Example

1. Show that the function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = 3x - 5$ is a bijective function.
2. Show that the function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^2$ is not bijective.

Example (A Function of Two Variables): Define a function $F : \mathbb{R} \times \mathbb{R} \to \mathbb{R} \times \mathbb{R}$ as :

$$\text{For all } (x, y) \in \mathbb{R} \times \mathbb{R}, F(x, y) = (x + y, x - y).$$

Is F a one-to-one correspondence from $\mathbb{R} \times \mathbb{R}$ to itself?

Inverse Functions

If F is a one-to-one correspondence from a set X to a set Y, then there is a function from Y to X that "undoes" the action of F. That is, it sends each element of Y back to the element of X that it came from. This function is called the inverse function for F.

Inverse Functions

Suppose $F : X \to Y$ is a one-to-one correspondence. I.e., suppose F is one-to-one and onto. Then there is a function $F^{-1} : Y \to X$ that is defined as follows:
Given any element $y \in Y$,
$F^{-1}(y) =$ the unique element $x \in X$ such that $F(x)$ equals y. I.e.,

$$\boxed{F^{-1}(y) = x \quad \Leftrightarrow \quad y = F(x).}$$

The function F^{-1} is called the **inverse function** for F.

Theorem

If X and Y are sets and $F : X \to Y$ is one-to-one and onto, then $F^{-1} : Y \to X$ is also one-to-one and onto.

Inverse Functions

Example The function $f : \mathbb{R} \to \mathbb{R}$ defined by the formula $f(x) = 4x - 1$ for all real numbers x was shown to be one-to-one in the previous examples. Find its inverse function.

Inverse Functions

Example

A function $F : \mathbb{R} \times \mathbb{R} \to \mathbb{R} \times \mathbb{R}$ defined as,
for all $(x, y) \in \mathbb{R} \to \mathbb{R}$, $F(x, y) = (x + y, x - y)$ is shown to be a bijective function in the previous example. Determine its inverse function $F^{-1} : \mathbb{R} \times \mathbb{R} \to \mathbb{R} \times \mathbb{R}$.