System Equations Estimation Methods

After finish this session, you should understand:

Finite Sample Asymptotic Relative.

- Unbiased vs Consistent vs Efficient
- The differences between Single vs System equations estimation methods
- Instrumental Variables (IV) technique
- OLS vs ILS vs 2SLS vs 3SLS
- Seemingly Unrelated Models
- Simultaneous Equations Models

System Equations Estimation Methods

System Equations Models

✓ Seemingly Unrelated (SUR) Model

✓ Simultaneous Equations Model

- Limited Information Estimation Methods (Single Equation Estimation Methods)
  - ILS, 2SLS, & (LIML)
- Full Information Estimation Methods (System Equation Estimation Methods)
  - 3SLS, I3SLS, & (FIML)

System Equations Estimation Methods

100 1,000 10,000 1,000,000

$n_1=10$ $\hat{\beta}_1$

$n_2=10$ $\hat{\beta}_2$

$n_{10,000}=10$ $\hat{\beta}_{10,000}$

Finite Sample (Small Sample) Properties

Unbiased $E(\hat{\beta})=\beta$ $E(\hat{\beta}) = \beta$ (All sizes of sample)

Asymptotic (Large Sample) Properties $n\to\infty$

Consistent $p\lim(\hat{\beta}) = \beta$ $n\to\infty$

$\lim\limits_{n\to\infty} \Pr(\hat{\beta} - \beta = \varepsilon) = 0$ where $\varepsilon \neq 0,\ \varepsilon \to 0$

(Only when $n \to \infty$) $\xrightarrow[n\to\infty]{}$ $\hat{\beta}=\beta$ $\underset{n\to\infty}{p\lim}\ \hat{\beta} = \beta$

Relative Properties

Efficient $\mathrm{var}(\hat{\beta}) < \mathrm{var}(\hat{\beta}_{other})$

OLS → BLUE

consistent $n\to\infty$

(Compare with …)

Seemingly Unrelated Regression (SUR) Models

The models:
$$\begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{bmatrix} = \begin{bmatrix} X_1 & 0 & \cdots & 0 \\ 0 & X_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & X_m \end{bmatrix} \begin{bmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_m \end{bmatrix} + \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \vdots \\ \varepsilon_m \end{bmatrix}$$

or $$y = X\beta + \varepsilon$$

where

$$E[\varepsilon] = 0, \operatorname{var}(\varepsilon) = \Sigma = \begin{bmatrix} \sigma_{11}I & \sigma_{12}I & \cdots & \sigma_{1m}I \\ \sigma_{21}I & \sigma_{22}I & \cdots & \sigma_{2m}I \\ \vdots & \vdots & \ddots & \vdots \\ \sigma_{m1}I & \sigma_{m2}I & \cdots & \sigma_{mm}I \end{bmatrix}$$

Seemingly Unrelated Regression (SUR) Models

The models:

$$
\begin{bmatrix}
\begin{bmatrix} y_{11} \\ y_{12} \\ \vdots \\ y_{1t_1} \end{bmatrix} \\
\begin{bmatrix} y_{21} \\ y_{22} \\ \vdots \\ y_{2t_2} \end{bmatrix} \\
\vdots \\
\begin{bmatrix} y_{m1} \\ y_{m2} \\ \vdots \\ y_{mt_m} \end{bmatrix}
\end{bmatrix}
=
\begin{bmatrix}
\begin{bmatrix} 1 & X_{111} & \cdots & X_{1k_1 1} \\ 1 & X_{112} & \cdots & X_{1k_1 2} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & X_{11t_1} & \cdots & X_{1k_1 t_1} \end{bmatrix} &
\begin{bmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix} &
\cdots &
\begin{bmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix} \\
\begin{bmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix} &
\begin{bmatrix} 1 & X_{211} & \cdots & X_{2k_2 1} \\ 1 & X_{212} & \cdots & X_{2k_2 2} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & X_{21t_2} & \cdots & X_{2k_2 t_2} \end{bmatrix} &
\cdots &
\begin{bmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix} \\
\vdots & \vdots & \ddots & \vdots \\
\begin{bmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix} &
\begin{bmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix} &
\cdots &
\begin{bmatrix} 1 & X_{m11} & \cdots & X_{mk_m 1} \\ 1 & X_{m12} & \cdots & X_{mk_m 2} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & X_{m1t_m} & \cdots & X_{mk_m t_m} \end{bmatrix}
\end{bmatrix}
\begin{bmatrix}
\begin{bmatrix} \beta_{10} \\ \beta_{11} \\ \vdots \\ \beta_{1k_1} \end{bmatrix} \\
\begin{bmatrix} \beta_{20} \\ \beta_{21} \\ \vdots \\ \beta_{2k_2} \end{bmatrix} \\
\vdots \\
\begin{bmatrix} \beta_{m0} \\ \beta_{m1} \\ \vdots \\ \beta_{mk_m} \end{bmatrix}
\end{bmatrix}
+
\begin{bmatrix}
\begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{12} \\ \vdots \\ \varepsilon_{1t_1} \end{bmatrix} \\
\begin{bmatrix} \varepsilon_{21} \\ \varepsilon_{22} \\ \vdots \\ \varepsilon_{2t_2} \end{bmatrix} \\
\vdots \\
\begin{bmatrix} \varepsilon_{m1} \\ \varepsilon_{m2} \\ \vdots \\ \varepsilon_{mt_m} \end{bmatrix}
\end{bmatrix}
$$

Seemingly Unrelated Regression (SUR) Models

$$\mathrm{var}(\varepsilon) = \Sigma = \begin{bmatrix} \begin{bmatrix} \sigma_1^2 & 0 & \cdots & 0 \\ 0 & \sigma_1^2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \sigma_1^2 \end{bmatrix} & \begin{bmatrix} \sigma_{12} & 0 & \cdots & 0 \\ 0 & \sigma_{12} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \sigma_{12} \end{bmatrix} & \cdots & \begin{bmatrix} \sigma_{1m} & 0 & \cdots & 0 \\ 0 & \sigma_{1m} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \sigma_{1m} \end{bmatrix} \\ \begin{bmatrix} \sigma_{21} & 0 & \cdots & 0 \\ 0 & \sigma_{21} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \sigma_{21} \end{bmatrix} & \begin{bmatrix} \sigma_2^2 & 0 & \cdots & 0 \\ 0 & \sigma_2^2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \sigma_2^2 \end{bmatrix} & \cdots & \begin{bmatrix} \sigma_{2m} & 0 & \cdots & 0 \\ 0 & \sigma_{2m} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \sigma_{2m} \end{bmatrix} \\ \vdots & \vdots & \ddots & \vdots \\ \begin{bmatrix} \sigma_{m1} & 0 & \cdots & 0 \\ 0 & \sigma_{m1} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \sigma_{m1} \end{bmatrix} & \begin{bmatrix} \sigma_{m2} & 0 & \cdots & 0 \\ 0 & \sigma_{m2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \sigma_{m2} \end{bmatrix} & \cdots & \begin{bmatrix} \sigma_m^2 & 0 & \cdots & 0 \\ 0 & \sigma_m^2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \sigma_m^2 \end{bmatrix} \end{bmatrix}$$

Seemingly Unrelated Regression (SUR) Models

reg y_1 x_1 x_2

reg y_2 x_2 x_3

Example: System of two equations:

(1)

$$y_{1t} = \beta_{10} + \beta_{11}x_{1t} + \beta_{12}x_{2t} + \varepsilon_{1t}$$
$$y_{2t} = \beta_{20} + \beta_{22}x_{2t} + \beta_{23}x_{3t} + \varepsilon_{2t}$$

$\hat{\varepsilon}_{1t}$, $\hat{\varepsilon}_{2t}$ $\Rightarrow \hat{\Sigma}$

If there exists relationship of the error terms across equations, i.e. $\boxed{E(\varepsilon_{1t}\varepsilon_{2t}) \neq 0}$.

Ignorance of this problem by using single equation estimation method (or separate OLS) will lead to less efficient estimators.

$\hat{\beta}$

$\sqrt{var(\hat{\beta})}\uparrow = S.E_{\hat{\beta}}\uparrow = $ t-test $\downarrow$

Seemingly Unrelated Regression (SUR) Models

Example: System of two equations:

$$\begin{array}{r} 1\\ 2\\ \vdots\\ 100\\ 101\\ 102\\ \vdots\\ 200 \end{array} \underbrace{\begin{bmatrix} \begin{bmatrix} y_{11}\\ y_{12}\\ \vdots\\ y_{1t} \end{bmatrix} \\ \begin{bmatrix} y_{21}\\ y_{22}\\ \vdots\\ y_{2t} \end{bmatrix} \end{bmatrix}}_{Y} = \underbrace{\begin{bmatrix} \begin{bmatrix} 1 & x_{11} & x_{21}\\ 1 & x_{12} & x_{22}\\ \vdots & \vdots & \vdots\\ 1 & x_{1t} & x_{2t} \end{bmatrix} & \begin{bmatrix} 0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0 \end{bmatrix} \\ \begin{bmatrix} 0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0 \end{bmatrix} & \begin{bmatrix} 1 & x_{21} & x_{31}\\ 1 & x_{22} & x_{32}\\ \vdots & \vdots & \vdots\\ 1 & x_{2t} & x_{3t} \end{bmatrix} \end{bmatrix}}_{X} \underbrace{\begin{bmatrix} \begin{bmatrix} \beta_{10}\\ \beta_{11}\\ \beta_{12} \end{bmatrix} \\ \begin{bmatrix} \beta_{20}\\ \beta_{22}\\ \beta_{23} \end{bmatrix} \end{bmatrix}}_{\beta} + \underbrace{\begin{bmatrix} \begin{bmatrix} \varepsilon_{11}\\ \varepsilon_{12}\\ \vdots\\ \varepsilon_{1t} \end{bmatrix} \\ \begin{bmatrix} \varepsilon_{21}\\ \varepsilon_{22}\\ \vdots\\ \varepsilon_{2t} \end{bmatrix} \end{bmatrix}}_{\varepsilon}$$

Seemingly Unrelated Regression (SUR) Models

If there exists relationship of the error terms across equations, i.e. $E(\varepsilon_{1t}\varepsilon_{2t}) = \sigma_{12} = \sigma_{21} \neq 0$

$$\mathrm{var}(\varepsilon) = \Sigma = \begin{bmatrix} \Sigma_1 & \Sigma_{12} \\ \Sigma_{21} & \Sigma_2 \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} \sigma_1^2 & 0 & \cdots & 0 \\ 0 & \sigma_1^2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \sigma_1^2 \end{bmatrix} & \begin{bmatrix} \sigma_{12} & 0 & \cdots & 0 \\ 0 & \sigma_{12} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \sigma_{12} \end{bmatrix} \\ \begin{bmatrix} \sigma_{21} & 0 & \cdots & 0 \\ 0 & \sigma_{21} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \sigma_{21} \end{bmatrix} & \begin{bmatrix} \sigma_2^2 & 0 & \cdots & 0 \\ 0 & \sigma_2^2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \sigma_2^2 \end{bmatrix} \end{bmatrix}$$

Estimation of SUR Models

Two-step system estimation or FGLS

1. Estimate Var-Cov matrix $\hat{\Sigma}$
2. Estimate all parameters β_j using FGLS.

$$\hat{\beta}_{SUR} = \hat{\beta}_{FGLS} = \left(X'\hat{\Sigma}^{-1}X\right)^{-1} X'\hat{\Sigma}^{-1}y$$

Then, system equation estimation method will provide a more asymptotically efficient estimators.

$\underset{n\to\infty}{Var}(\hat{\beta}_{FGLS}) < Var(\hat{\beta}_{OLS})$

However, if there exists specification error problem in any equations, the problem will spread through out the whole system.

Simultaneous Equations Model

More than one equation.

Jointly dependent or endogenous variables.

General Structural Form model can be stated as:

Exogeneous.

$$\gamma_{11}y_{t1}+\gamma_{21}y_{t2}+\cdots+\gamma_{m1}y_{tm}+\beta_{11}x_{t1}+\cdots+\beta_{k1}x_{tk}=\varepsilon_{t1}$$

$$\gamma_{12}y_{t1}+\gamma_{22}y_{t2}+\cdots+\gamma_{m2}y_{tm}+\beta_{12}x_{t1}+\cdots+\beta_{k2}x_{tk}=\varepsilon_{t2}$$

$$\vdots$$

$$\gamma_{1m}y_{t1}+\gamma_{2m}y_{t2}+\cdots+\gamma_{mm}y_{tm}+\beta_{1m}x_{t1}+\cdots+\beta_{km}x_{tk}=\varepsilon_{tm}$$

Simultaneous Equations Model

Structural Form model:

$$\begin{bmatrix} y_1 & y_2 & \cdots & y_m \end{bmatrix}_t \begin{bmatrix} \gamma_{11} & \gamma_{12} & \cdots & \gamma_{1m} \\ \gamma_{21} & \gamma_{22} & \cdots & \gamma_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ \gamma_{m1} & \gamma_{m2} & \cdots & \gamma_{mm} \end{bmatrix} +$$

$$\begin{bmatrix} x_1 & x_2 & \cdots & x_m \end{bmatrix}_t \begin{bmatrix} \beta_{11} & \beta_{12} & \cdots & \beta_{1m} \\ \beta_{21} & \beta_{22} & \cdots & \beta_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ \beta_{m1} & \beta_{m2} & \cdots & \beta_{mm} \end{bmatrix} = \begin{bmatrix} \varepsilon_{1t} & \varepsilon_{2t} & \cdots & \varepsilon_{mt} \end{bmatrix}$$

or

$$\mathrm{y}_t'\Gamma + \mathrm{x}_t'\mathrm{B} = \varepsilon_t'$$

Simultaneous ~~Equations~~ Bias. Model

Two endogenous variables models:

Endogeneity Bias.

$$y_{1t} = \beta_{10} + \beta_{12} y_{2t} + \gamma_{11} x_{1t} + \varepsilon_{1t}$$

$$y_{2t} = \beta_{20} + \beta_{21} y_{1t} + \gamma_{21} x_{1t} + \varepsilon_{2t}$$

$E(\varepsilon_{1t} y_{2t}) \neq 0$

$E(\varepsilon_{2t}, y_{1t}) \neq 0$

Example

Keynesian Income Determination Model

Consumption Function $\quad C_t = \beta_0 + \beta_1 Y_t + \varepsilon_{1t}$

Income Identity $\quad Y_t = C_t + I_t$

$E(\varepsilon_{1t} \varepsilon_{2t}) \neq 0$

System Equations Model

Simultaneous Bias occurs when $E(y_t \varepsilon_{1t}) \neq 0$

OLS estimators will be biased, inconsistent, and inefficient.

$E(\hat{\beta}) \neq \beta$ Biased

$p\lim(\hat{\beta}) \neq \beta$ Inconsistent

System Equations Model

$Y_t = \beta_0 + \beta_1 Y_t + I_t + \varepsilon_{1t}$

$Y_t - \beta_1 Y_t = \beta_0 + I_t + \varepsilon_{1t}$

$Y_t = \frac{\beta_0}{1-\beta_1} + \frac{I_t}{1-\beta_1} + \frac{\varepsilon_{1t}}{1-\beta_1}$

$Y_t = \Pi_2 + \Pi_3 I_t + w_{2t}$

Structural Form Model

shows structural relationship between endogenous variables and exogenous variables & lagged endogenous variables.

Theory

(2)

$$C_t = \beta_0 + \beta_1 Y_t + \varepsilon_{1t}$$

Identity

$$Y_t \equiv C_t + I_t$$

$C_t = \beta_0 + \beta_1 (C_t + I_t) + \varepsilon_{1t}$

$C_t - \beta_1 C_t = \beta_0 + \beta_1 I_t + \varepsilon_{1t}$

$(1-\beta_1) C_t = \beta_0 + \beta_1 I_t + \varepsilon_{1t}$

$C_t = \frac{\beta_0}{(1-\beta_1)} + \frac{\beta_1}{(1-\beta_1)} I_t + \frac{\varepsilon_{1t}}{(1-\beta_1)}$

$C_t = \Pi_0 + \Pi_1 I_t + w_{1t}$

Reduced Form Model

shows relationship only between endogenous variables and exogenous variables.

(1)

$$C_t = \pi_0 + \pi_1 I_t + w_{1t}$$

$$Y_t = \pi_2 + \pi_3 I_t + w_{2t}$$

where

$$\pi_0 = \frac{\beta_0}{1-\beta_1} \qquad \pi_1 = \frac{\beta_1}{1-\beta_1}$$

$$\pi_2 = \frac{\beta_0}{1-\beta_1} \qquad \pi_3 = \frac{1}{1-\beta_1}$$

$$w_{1t} = \frac{\varepsilon_{1t}}{1-\beta_1} \qquad w_{2t} = \frac{\varepsilon_{1t}}{1-\beta_1}$$

System Equations Model

<u>Identification problem</u>:

√ - Over-identified system   3 eq. & 2 unk. → more than 1 sol.

√ - Exactly-identified system   2 eq & 2 unk. → unique solution.

√ - Under-identified system   2 eq. & 3 unk. → can't solve sol. no. sol.

Single Equation Estimation Methods

<u>Indirect Least Squares (ILS)</u>

unique solution

Only for exactly-identified system

- Estimate Reduced form model using OLS
- Solve for estimated β

Structural Form Model:

$$Y\Gamma + XB = E$$

Reduced Form Model:

$$Y = X\hat{\Pi} + V$$

ILS: 1. Estimate Reduced Form $\hat{\Pi} = (X'X)^{-1} X'Y$

2. Solve for B $\hat{B}_{ILS} = -\hat{\Pi}\Gamma$

Single Equation Estimation Methods

<u>Two-Stage Least Squares (2SLS)</u> more than 1 sol.

For both Exactly and Over-identified systems

1. Estimate Reduced form model using OLS

   Predict endogenous variable ($\hat{Y}$)

2. Use $\hat{Y}$ as Instrumental Variables (IV) for lagged endogenous variables instead of using actual Y, then, obtain matrix $\hat{Z}$, then estimate model using IV technique

$$\hat{B} = \left(\hat{Z}'\hat{Z}\right)^{-1} \hat{Z}'Y$$

System Equations Estimation Methods

Three-Stage Least Squares (3SLS)

If there exists relationship of the error terms across equations, system equation estimation will lead to more asymptotically efficient estimators.

2SLS + construct estimated Σ matrix

3-stage perform FGLS using estimated $\hat{\Sigma}$ matrix

$$\boxed{\hat{B} = \left[\hat{Z}'\left(\hat{\Sigma}^{-1}\right)\hat{Z}\right]^{-1}\hat{Z}'\left(\hat{\Sigma}^{-1}\right)y}$$

OLS, ILS, 2SLS, 3SLS

1. OLS
2. IV
3. FGLS

Whole system

Run One eq. at a time

1 Single Equation Estimation Methods

OLS: biased, inconsistent, & inefficient

ILS: biased, consistent, & Asymptotically efficient

2SLS: biased, consistent, & Asymptotically efficient

Run all eq (models) in 1 time.

2 System Equation Estimation Methods

~~3SLS:~~ biased, consistent?, & More Asymptotically efficient if there exists correlation of error terms across eq.

★★★ but if there exists specification error in one or any equation(s), the specification error problem will spread through all equations in the system → Inconsistent.

Specification Test – Hausman Test

Hypothesis OLS estimators are consistent compared to alternative method estimators.

Reject H_0 means there is exists endogeneity (simultaneous) bias (or x's are endogenous).

Fail to Reject H_0 means there is no endogeneity (simultaneous) bias (or x's are exogenous).