

CHAPTER 4

Basic Matrix Algebra and Applications

Topics: **Basic Matrix Algebra**

Outline:

- ☺ Representation of system of equation by matrix notation
- ☺ Multiplication of matrices
- ☺ Determinant and singularity of matrix
- ☺ Matrix Inversion
- ☺ Cramer's rule

Matrix Algebra

Matrix is a rectangular array of numbers, parameters, or variables

$$\begin{bmatrix} \text{☺} & \text{☺} & \text{☺} & \dots & \text{☺} \\ \text{☺} & \text{☺} & \text{☺} & \dots & \text{☺} \\ \cdot & \cdot & \cdot & \dots & \cdot \\ \cdot & \cdot & \cdot & \dots & \cdot \\ \text{☺} & \text{☺} & \text{☺} & \dots & \text{☺} \end{bmatrix}$$

$$\begin{bmatrix} \spadesuit & \heartsuit \\ \clubsuit & \diamondsuit \end{bmatrix}, \begin{pmatrix} \spadesuit & \heartsuit \\ \clubsuit & \diamondsuit \end{pmatrix}, \left\| \begin{matrix} \spadesuit & \heartsuit \\ \clubsuit & \diamondsuit \end{matrix} \right\|$$

$$\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}_{m \times n}$$

$$[a_{ij}]_{m \times n}, i = 1, \dots, m; j = 1, \dots, n$$

Benefits of using Matrix

1. Matrix provides a compact way of writing an equation system, even an extremely large one
2. Matrix provides a way of testing the existence of a solution by evaluation of a determinant.
3. Matrix can be used to find that solution if exists. This is done by using Inverse matrix or Cramer's rule

Limitation: matrix algebra is applicable only to linear-equation systems.

In some cases, we can transform variables so as to obtain a linear relation to work with. For example, the nonlinear function $y = ax^b$ can be transformed into:

$$\log(y) = \log(a) + b \log(x)$$

A compact way of writing an equation system:

A system of m linear equations with n endogenous variables:

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= d_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= d_2 \\ \vdots &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= d_m \end{aligned}$$

In matrix form:

$$Ax = d$$

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}_{m \times n} \quad x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}_{n \times 1} \quad d = \begin{bmatrix} d_1 \\ d_2 \\ \vdots \\ d_m \end{bmatrix}_{m \times 1}$$

Note: Terminology

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}_{n \times 1} : \text{column vector}$$

$$z = (z_1 \quad z_2 \quad \cdots \quad z_n)_{1 \times n} : \text{row vector}$$

Recapping some Matrix definitions

Definition 1

Two matrices $A = [a_{ij}] \in \mathbb{R}^{m \times n}$ and $B = [b_{ij}] \in \mathbb{R}^{p \times q}$ are equal, written $A = B$ if $m=p$ and $n=q$, and $a_{ij} = b_{ij}$ for all i and j

Example $A = \begin{bmatrix} 4 & 2 \\ 1 & 3 \end{bmatrix} \quad B = \begin{bmatrix} 4 & 2 \\ 3 & 1 \end{bmatrix} \quad C = \begin{bmatrix} 4 & 2 \\ 1 & 3 \end{bmatrix}$

Definition 2

Given a matrix $A = [a_{ij}] \in \mathbb{R}^{m \times n}$ and a scalar $c \in \mathbb{R}$
 the **scalar multiplication of matrix A** by
 the scalar c is given by $cA = [ca_{ij}] \in \mathbb{R}^{m \times n}$

Example

$$7 \begin{bmatrix} 3 & -1 \\ 0 & 5 \end{bmatrix} =$$

Definition 3

The **addition of two matrices** $A = [a_{ij}] \in \mathbb{R}^{m \times n}$
 and $B = [b_{ij}] \in \mathbb{R}^{p \times q}$ is defined only if $m=p$
 And $n=q$, and is given by

$$A + B = [a_{ij}] + [b_{ij}] = [a_{ij} + b_{ij}] \in \mathbb{R}^{m \times n}$$

Example

$$\begin{bmatrix} 4 & 9 \\ 2 & 1 \end{bmatrix} + \begin{bmatrix} 2 & 0 \\ 0 & 7 \end{bmatrix} =$$

Transposition of a Matrix**Definition 6**

Let $A = [a_{ij}] \in \mathbb{R}^{m \times n}$, The transpose of A is
 given by,

$$A^T = [a_{ij}^T] \in \mathbb{R}^{n \times m}, \text{ where } a_{ij}^T = a_{ji}$$

When a matrix is transposed, the rows becomes
 columns and vice versa

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{pmatrix}_{2 \times 3}$$

$$A^T = \begin{pmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \\ a_{13} & a_{23} \end{pmatrix}_{3 \times 2}$$

Example

$$A = \begin{bmatrix} 3 & 8 & 9 \\ 1 & 0 & 4 \end{bmatrix} \quad A' =$$

$$B = \begin{bmatrix} 3 & 4 \\ 1 & 7 \end{bmatrix} \quad B' =$$

$$(AB)^T = B^T A^T$$

Recapping some special types of matrices

>> Square matrix

A matrix $A = [a_{ij}] \in \mathbb{R}^{m \times n}$ is a **square matrix** if $m=n$. That is, the number of rows is equal to the number of columns

Example

$$A = \begin{bmatrix} 1 & 3 & 4 \\ 2 & 4 & 5 \\ 5 & 2 & 9 \end{bmatrix} \quad B = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix}$$

>> Diagonal matrix

The **diagonal elements** of a square matrix

$A \in \mathbb{R}^{m \times m}$ are the elements

$$a_{11}, a_{22}, \dots, a_{mm}$$

A **diagonal matrix** is a square matrix

$$A = [a_{ij}] \in \mathbb{R}^{n \times n}$$

where $a_{ij} = 0$ if $i \neq j$

$$\text{Let } I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad B = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

>> Identity matrix

An **identity matrix** is a diagonal matrix I_n of order $n \times n$ whose diagonal elements being ones. An identity matrix is thus necessarily square.

$$I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}_{2 \times 2} \quad I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}_{3 \times 3}$$

>> Null matrix

A matrix $A = [a_{ij}] \in \mathbb{R}^{m \times n}$ is a **zero matrix/null matrix** if all of its elements are zeros. We write

$$A = 0$$

Example

$$A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

>>Triangular matrix

A matrix “A” is **upper triangular** if it is square and $a_{ij} = 0$ if $i > j$, and **lower triangular** if it is square and $a_{ij} = 0$ if $i < j$

$$L = \begin{bmatrix} 2 & 0 & 0 \\ 1 & 3 & 0 \\ 6 & 5 & 4 \end{bmatrix} \quad U = \begin{bmatrix} 2 & 1 & 6 \\ 0 & 3 & 5 \\ 0 & 0 & 4 \end{bmatrix}$$

>> Symmetric matrix

A matrix is symmetric if $A^T = A$. That is $a_{ij} = a_{ji}$ for all i, j when

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{bmatrix}$$

We will learn the following matrix operation (Matrix Algebra)

↗ Multiplication of matrices

↗ Determinant of a matrix

↗ Inverses

↗ Multiplication of matrices

The Multiplication of matrix $A = [a_{ij}]_{m \times n}$ by matrix $B = [b_{ij}]_{p \times q}$ is defined only if

$n = p$ and is given by $AB = \left[\sum_{k=1}^n a_{ik} b_{kj} \right]_{m \times q}$

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Example

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \end{bmatrix} = \begin{bmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{bmatrix}$$

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \end{bmatrix} = \begin{bmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{bmatrix}$$

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \end{bmatrix} = \begin{bmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{bmatrix}$$

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} 5 & 6 \\ 7 & 8 \end{pmatrix} =$$

Question: $A_{4 \times 3} B_{2 \times 4} = ?$

“dot product”

Definition 5

When a row vector is multiplied by a column vector, the product is called **dot product or inner product**. That is, if

$$\mathbf{a} \mathbf{b} = [a_1 \quad a_2 \quad \dots \quad a_n] \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix} = \left[\sum_{i=1}^n a_i b_i \right] \in \mathbf{R}^{1 \times 1} = \mathbf{R},$$

$$u = [3 \quad 4]_{1 \times 2} \quad v = \begin{bmatrix} 9 \\ 7 \end{bmatrix}_{2 \times 1} \quad uv = [3*9 + 4*7] = [55] = 55$$

Note: A matrix is idempotent if $A^2 = A$

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$$A = \begin{pmatrix} 5 & -5 \\ 4 & -4 \end{pmatrix}$$

➤ Determinant

The determinant of a square matrix A , denoted by $|A|$, is a **uniquely defined number** associated with that matrix

For 2 by 2 matrix $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$, the 2nd order determinant is

$$|A| = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = \dots\dots\dots$$

$$|A| = \begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} =$$

Evaluating a Third-Order Determinant

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}, \text{ its determinant is}$$

$$|A| = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

$$= a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{31}a_{22}a_{13} - a_{32}a_{23}a_{11} - a_{33}a_{21}a_{12}$$

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} =$$

n^{th} -order determinant by Laplace Expansion

Let A be a matrix in $R^{n \times n}$. The determinant of A or $\det A$ is given by $|A|$.

$$|A| = \begin{cases} \sum_{j=1}^n a_{ij} C_{ij} & i = 1, 2, \dots, n \text{ (expansion by } i^{\text{th}} \text{ row)} \\ \sum_{i=1}^n a_{ij} C_{ij} & j = 1, 2, \dots, n \text{ (expansion by } j^{\text{th}} \text{ column)} \end{cases}$$

If we choose row i ,

$$\begin{bmatrix} a_{11} & a_{12} & a_{1j} & a_{1n} \\ a_{21} & a_{22} & a_{2j} & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{i1} & a_{i2} & a_{ij} & a_{in} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & a_{nj} & a_{nn} \end{bmatrix}_{n \times n}$$

If we choose column j ,

$$\begin{bmatrix} a_{11} & a_{12} & a_{1j} & a_{1n} \\ a_{21} & a_{22} & a_{2j} & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{i1} & a_{i2} & a_{ij} & a_{in} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & a_{nj} & a_{nn} \end{bmatrix}_{n \times n}$$

>> We call C_{ij} , Cofactor of *element* a_{ij} .

>> C_{ij} can be found by $(-1)^{i+j} M_{ij}$

The minor of element a_{ij} , M_{ij} , is the determinant of the matrix resulting from deleting row i and column j of matrix $A_{n \times n}$. In other words, a "minor" is the determinant of the square matrix formed by deleting one row and one column from some larger square matrix.

Example, find the following determinant:

$$\begin{vmatrix} 15 & 7 & 9 \\ 2 & 5 & 6 \\ 9 & 0 & 12 \end{vmatrix}$$

Properties of Determinants

Property I

The interchange of rows and columns does not affect the value of a determinant. In other words, the determinant of a matrix A has the same value as that of its transpose A' , that is

Example

$$A = \begin{bmatrix} 4 & 3 \\ 5 & 6 \end{bmatrix} \quad \det A =$$

$$B = \begin{bmatrix} 4 & 5 \\ 3 & 6 \end{bmatrix} \quad \det B =$$

Property II

The interchange of any two rows (or any two columns) will alter the sign, but not the numerical value, of the determinant

Example

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad \det A =$$

$$B = \begin{bmatrix} c & d \\ a & b \end{bmatrix} \quad \det B =$$

Property III

The multiplication of any one row (or one column) by a scalar k will change the value of the determinant k -fold

Example

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad \det A =$$

$$B = \begin{bmatrix} ka & kb \\ c & d \end{bmatrix} \quad \det B =$$

Property IV

The addition (subtraction) of a multiple of any row to (from) another row will leave the value of the determinant unaltered

Example

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad \det A =$$

$$B = \begin{bmatrix} a & b \\ c+ka & d+kb \end{bmatrix} \quad \det B =$$

Property V

If one row (or column) is a multiple of another row (or column), the value of the determinant will be zero

Example

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad \det A =$$

$$B = \begin{bmatrix} 2a & 2b \\ a & b \end{bmatrix} \quad \det B =$$

The determinant & Non-singularity

If $|A| = 0$, square matrix $A_{n \times n}$ is a singular Matrix

: A square matrix that is not invertible is called **singular** or **degenerate**. A square matrix is singular if and only if its determinant is 0

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If $|A| \neq 0$, matrix A is. a non-singular Matrix

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↗ The inverse of A

Assume that an $n \times n$ nonsingular matrix A is given by:

$$\begin{bmatrix} a_{11} & a_{12} & a_{1j} & a_{1n} \\ a_{21} & a_{22} & a_{2j} & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{i1} & a_{i2} & a_{ij} & a_{in} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & a_{nj} & a_{nn} \end{bmatrix}_{n \times n}$$

$$A^{-1} = \frac{1}{|A|} \cdot \text{adj}A$$

To find the inverse of matrix A :

- 1.) Find $\det A, |A|$, if $|A| = 0$, A is a singular matrix and the inverse of matrix A cannot be found.
- 2.) Find Minor (M_{ij}) for all element of the matrix in row $i = 1, \dots, n$ and column $j = 1, \dots, n$
- 3.) Find Cofactor (C_{ij}) from Minor (M_{ij}) for all elements of the matrix in row $i = 1, \dots, n$ and column $j = 1, \dots, n$
- 4.) Get the Cofactor matrix C in which each element is Cofactor (C_{ij}) of element a_{ij}
- 5.) The transpose of C is called Adjoint A ($\text{adj}A$)

$$\text{adj}A = [C_{ij}]^T$$

- 6.) Multiple $\text{adj}A$ by scalar $\frac{1}{\det A}$ $\text{adj}A$ and we will get the inverse of matrix A , A^{-1} .

Properties of the inverse matrix A^{-1}

- I.....
- II.....
- III.....
- IV.....

Find A^{-1}

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$A = \begin{bmatrix} 3 & 2 \\ 1 & 0 \end{bmatrix}$$

$$A = \begin{pmatrix} 15 & 7 & 9 \\ 2 & 5 & 6 \\ 9 & 0 & 12 \end{pmatrix}$$

To find the solution for system of equation by using matrix

We can use (1.) Inverse matrix (2.) Cramer's rule

Remind that:

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= d_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= d_2 \\ \vdots &\quad \quad \quad \vdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n &= d_n \end{aligned}$$

We can write this system of equations in the form of matrix as:

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To find solution to linear system of equations: $A_{n \times n} x_{n \times 1} = d_{n \times 1}$ CHECK IF DETERMINANT OF MATRIX A IS NOT EQUAL TO ZERO.			
Vector d		$d_{n \times 1} \neq 0$	$d_{n \times 1} = 0$ (homogenous equation system)
Det A			
$ A \neq 0$ (Linear independent eq) Matrix A is non-singular A^{-1} exist		Unique solution $x = A^{-1}d \neq \underline{0}$ e.g. $\begin{bmatrix} 5 & 3 \\ 6 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 30 \\ 8 \end{bmatrix}$	Unique solution $x = A^{-1}\underline{0} = \underline{0}$ e.g. $\begin{bmatrix} 5 & 3 \\ 6 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$
$ A = 0$ Matrix A singular A^{-1} does not exist	Linear dependent eq. $ A_k = 0, A = 0$	Infinite solutions (not including $x = \underline{0}$) e.g. $\begin{bmatrix} 5 & 3 \\ 10 & 6 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 30 \\ 60 \end{bmatrix}$	Infinite solutions (including $x = \underline{0}$) e.g. $\begin{bmatrix} 5 & 3 \\ 10 & 6 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$
	Inconsistent eq. $ A_k \neq 0, A = 0$	No solution $\begin{bmatrix} 5 & 3 \\ 10 & 6 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 30 \\ 65 \end{bmatrix}$	n/a

Exercise: Consider the following system of equations and examine whether there is(are) solution(s) to the system or not.

1.)	$4x + y + 2z = 5$ $5x + 2y + z = 7$ $x + z = 1$	2.)	$3x = 12 - 5y + 4z$ $6x = 9 + 10y$ $8x = 10 + 7y - 3z$
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>> Use inverse matrix

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>> Use Cramer's Rule:

From,

$$X_{n \times 1} = A_{n \times n}^{-1} d_{n \times 1}$$

That is,

$$X = \frac{1}{|A|} \begin{pmatrix} C_{11} & C_{12} & \cdots & C_{1n} \\ C_{21} & C_{22} & \cdots & C_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ C_{n1} & C_{n2} & \cdots & C_{nn} \end{pmatrix}^T \begin{bmatrix} d_1 \\ \vdots \\ d_n \end{bmatrix}$$

$$; C_{ij} = (-1)^{i+j} M_{ij} \quad \text{and}$$

$$|A| = \begin{cases} \sum_{j=1}^n a_{ij} C_{ij} & i = 1, 2, \dots, n \text{ (expansion by } i^{\text{th}} \text{ row)} \\ \sum_{i=1}^n a_{ij} C_{ij} & j = 1, 2, \dots, n \text{ (expansion by } j^{\text{th}} \text{ column)} \end{cases}$$

$$X = \frac{1}{|A|} \begin{pmatrix} C_{11} & C_{12} & \cdots & C_{1n} \\ C_{21} & C_{22} & \cdots & C_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ C_{n1} & C_{n2} & \cdots & C_{nn} \end{pmatrix} \begin{bmatrix} d_1 \\ \vdots \\ d_n \end{bmatrix}$$

$$X = \begin{pmatrix} \frac{1}{|A|} (C_{11}d_1 + C_{12}d_2 + \cdots + C_{1n}d_n) \\ \frac{1}{|A|} (C_{21}d_1 + C_{22}d_2 + \cdots + C_{2n}d_n) \\ \vdots \\ \frac{1}{|A|} (C_{n1}d_1 + C_{n2}d_2 + \cdots + C_{nn}d_n) \end{pmatrix} = \begin{pmatrix} \frac{1}{|A|} \sum_{i=1}^n C_{i1}d_i \\ \frac{1}{|A|} \sum_{i=1}^n C_{i2}d_i \\ \vdots \\ \frac{1}{|A|} \sum_{i=1}^n C_{in}d_i \end{pmatrix}$$

$C_{11}d_1 + C_{21}d_2 + \cdots + C_{n1}d_n$ is the determinant of matrix A when replacing column 1 of matrix A by vector d . That is,

$$C_{11}d_1 + C_{21}d_2 + \cdots + C_{n1}d_n = \begin{vmatrix} d_1 & a_{12} & \cdots & a_{1n} \\ d_2 & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ d_n & a_{n2} & \cdots & a_{nn} \end{vmatrix} = |A_1|$$

Likewise,

$$\sum_{i=1}^n C_{i2}d_i = \begin{vmatrix} a_{11} & d_1 & \cdots & a_{1n} \\ a_{21} & d_2 & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & d_n & \cdots & a_{nn} \end{vmatrix} = |A_2|$$

$$\sum_{i=1}^n C_{in}d_i = \begin{vmatrix} a_{11} & a_{12} & \cdots & d_1 \\ a_{21} & a_{22} & \cdots & d_2 \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & d_n \end{vmatrix} = |A_n|$$

Therefore, Cramer's rule is

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