Macroeconomics

Lecture 3

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Bellman's Equations

- Define the (T+1)-period value function

$$W_{T+1}(x_0) = \max_{u_0,\dots,u_T} \left[r_0(x_0,u_0) + r_1(x_1,u_1) + \dots + r_T(x_T,u_T) + W_0(x_{T+1}) \right] \qquad (1)$$

$$\textit{Subject to} \qquad x_{t+1} = g_t(x_t,u_t), \qquad t = 0,1,\dots.; \quad x_0 \ \textit{given.}$$

Problem (1) is the same as

$$W_{T+1}(x_0) = \max_{u_0} \left[r_0(x_0,u_0) + \max_{u_1} \left[r_1(x_1,u_1) + \dots \max_{u_T} \left[r_T(x_T,u_T) + W_0(x_{T+1}) \right] \right] \right] \quad (2)$$

$$\textit{Subject to} \quad x_{t+1} = g_t(x_t,u_t), \qquad x_0 \ \textit{given.}$$

(1) Solving for $u_T=h_T(x_T)$ to optimize $W_1(x_T)$,

(2) Solving for $u_{T-1}=h_{T-1}(x_{T-1})$ to optimize $W_2(x_{T-1})$,

Continue to repeat this process until t=0.

Self-enforcing property

- The optimal policies $u_t=h_t(x_t)$, t=0,1,2…,T, from problem (1) will have the same values of

  $u_s=h_s(x_s)$, t=s,s+1,s+2…,T,

  as those optimal policies obtained from the following problem: for s>0

$$\max_{u_s,\ldots,u_T}\left[r_s\left(x_s,u_s\right)+r_{s+1}\left(x_{s+1},u_{s+1}\right)+\ldots+r_T\left(x_T,u_T\right)+W_0\left(x_{T+1}\right)\right] \qquad (3)$$

$$Subject\ to \qquad x_{t+1}=g_t\left(x_t,u_t\right), \qquad t=s,s+1,\ldots,; \quad x_s \ given.$$

- Then, these optimal policies are said to be “time consistent”.

Policy function that does not vary over time

- Assume that: $r_t(x_t,u_t) = \beta^t.r(x_t,u_t)$, $0<\beta<1$
- $g_t(x_t,u_t) = g(x_t,u_t)$, (4)
- Bellman's equation becomes

$$W_{j+1}\left(x_{T-j}\right)=\max_{u_{T-j}}\left[\beta^{T-j}r\left(x_{T-j},u_{T-j}\right)+W_j\left(x_{T-j+1}\right)\right]$$

$$\beta^{j-T}W_{j+1}\left(x_{T-j}\right)=\max_{u_{T-j}}\left[r\left(x_{T-j},u_{T-j}\right)+\beta.\beta^{j-T-1}W_j\left(x_{T-j+1}\right)\right]$$

$$V_{j+1}\left(x_{T-j}\right)=\max_{u_{T-j}}\left[r\left(x_{T-j},u_{T-j}\right)+\beta.V_j\left(x_{T-j+1}\right)\right] \qquad (5)$$

- where

$$V_{j+1}\left(x_{T-j}\right)=\beta^{j-T}W_{j+1}\left(x_{T-j}\right)$$

$$If\ \ j=T,\ then \qquad V_{T+1}\left(x_0\right)=W_{T+1}\left(x_0\right)$$

Policy function that does not vary over time

$$V_{j+1}\left(x_{T-j}\right)=\max_{u_{T-j},\ldots,u_T}\left[r\left(x_{T-j},u_{T-j}\right)+\beta r\left(x_{T-j+1},u_{T-j+1}\right)+\ldots+\beta^{j} r\left(x_T,u_T\right)+\beta^{j+1}V_0\left(x_{T+1}\right)\right]$$

- The Bellman's equation becomes

$$V_{j+1}\left(x_{T-j}\right)=\max_{u_{T-j}}\left[r\left(x_{T-j},u_{T-j}\right)+\beta V_j\left(x_{T-j+1}\right)\right] \qquad (6)$$

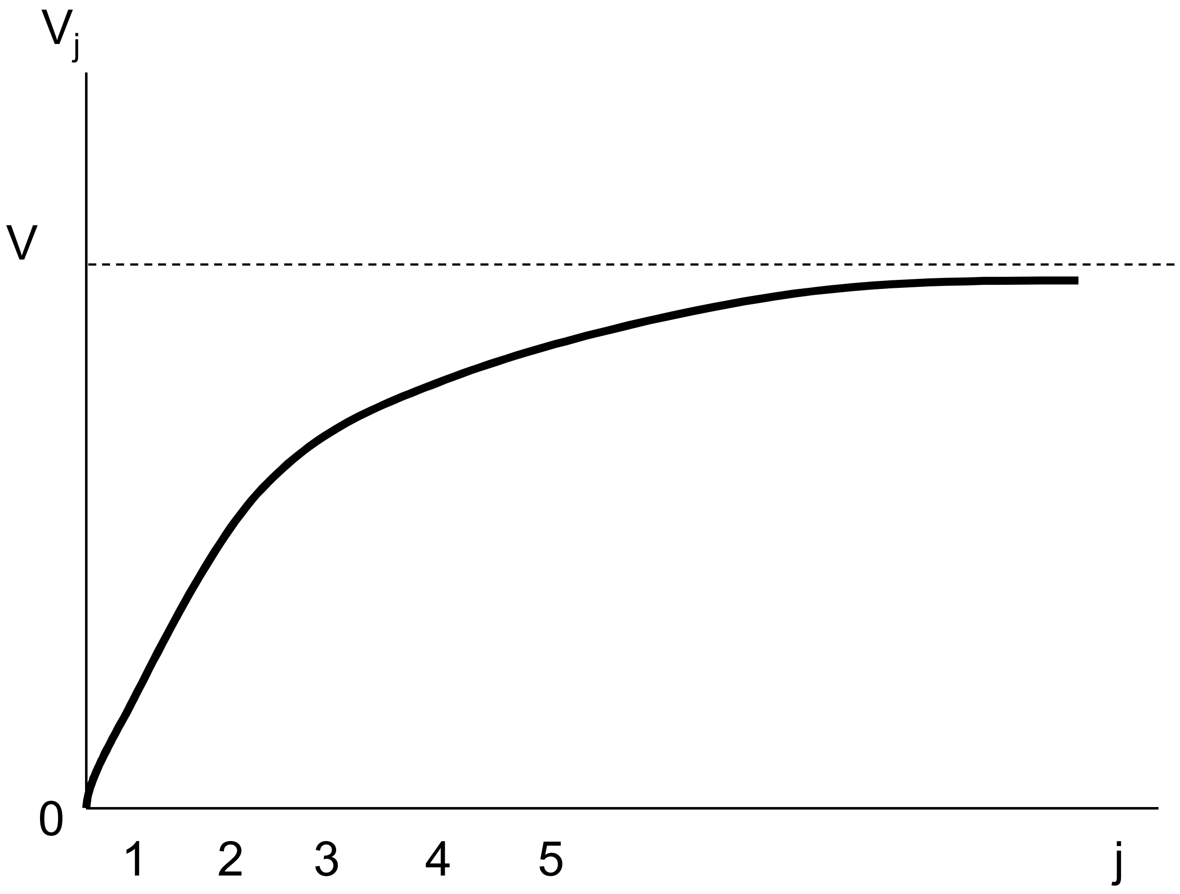
$$\textit{Subject to} \qquad x_{T-j+1}=g\left(x_{T-j},u_{T-j}\right), \quad x_{T-j} \ \textit{given}$$

- In the case that $V=\lim_{j\to\infty} V_j$, then

$$V\left(x\right)=\max_{u}\left[r\left(x,u\right)+\beta V\left(\tilde{x}\right)\right] \qquad (7)$$

$$\textit{Subject to} \qquad \tilde{x}=g\left(x,u\right), \quad x \ \textit{given.}$$

Policy function that does not vary over time



Discounted Dynamic Programming Problem

- The limiting value function V that solves (7) is the optimal value function for the infinite horizon problem

$$V(x_0) = \max_{\{u_t\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t r(x_t, u_t) \qquad (8)$$

$$\textit{Subject to} \quad x_{t+1} = g(x_t, u_t), \qquad x_0 \ \textit{given}$$

- There is a unique and time-invariant optimal policy
- $u_t=h(x_t)$, t=0,1,2…,T
- Where h(.) is chosen to maximize the right hand side of (7), and

Discounted Dynamic Programming Problem

the limiting value function V is differentiable with respect to x. (This is the same as the marginal condition (1.4b) in Lecture #1). Hence,

$$V'(x) = \frac{\partial r\left[x, h(x)\right]}{\partial x} + \beta \frac{\partial g\left[x, h(x)\right]}{\partial x} V'\left(g\left[x, h(x)\right]\right) \qquad (9)$$

Example

$$\max \quad \sum_{t=0}^{\infty} \beta^t U(c_t), \qquad 0<\beta<1$$

$$Subject\ \ to$$

$$A_{t+1} = R_t\left(A_t + y_t - c_t\right),$$

$$or,\ \ c_t + \sum_{j=1}^{\infty}\left(\prod_{k=0}^{j-1} R_{t+k}^{-1}\right)c_{t+j} = y_t + \sum_{j=1}^{\infty}\left(\prod_{k=0}^{j-1} R_{t+k}^{-1}\right)y_{t+j} + A_t. \qquad (10)$$

$$A_0\ \ given,$$

- State variables are A_t , y_t, R_{t-1}.
- Control variables is $u_t = A_t + y_t - c_t$
- Transition function $A_{t+1} = R_t u_t$ does not involve state variable. (Or in this case, we have $x_{t+1}=g(u_t)$, instead of $x_{t+1}=g(x_t, u_t)$)

$$A_{t+1} = R_t\left(A_t + y_t - c_t\right) \qquad (a)$$

$$A_{t+2} = R_{t+1}\left(A_{t+1} + y_{t+1} - c_{t+1}\right) \qquad (b)$$

$$A_{t+3} = R_{t+2}\left(A_{t+2} + y_{t+2} - c_{t+2}\right) \qquad (c)$$

Put (a) into (b)

$$A_{t+2} = R_{t+1}\left(R_t\left[A_t + y_t - c_t\right] + y_{t+1} - c_{t+1}\right) \qquad (d)$$

Put (d) into (c)

$$A_{t+3} = R_{t+2}\left(R_{t+1}\left(R_t\left[A_t + y_t - c_t\right] + y_{t+1} - c_{t+1}\right) + y_{t+2} - c_{t+2}\right)$$

$$A_{t+3} = R_{t+2}R_{t+1}R_t\left[A_t + y_t - c_t\right] + R_{t+2}R_{t+1}\left[y_{t+1} - c_{t+1}\right] + R_{t+2}\left[y_{t+2} - c_{t+2}\right]$$

$$A_{t+3} = \left[\prod_{k=0}^{2} R_{t+k}\right]\left[A_t + y_t - c_t\right] + \left[\prod_{k=1}^{2} R_{t+k}\right]\left[y_{t+1} - c_{t+1}\right] + R_{t+2}\left[y_{t+2} - c_{t+2}\right]$$

$$A_{t+j} = \left[\prod_{k=0}^{j-1} R_{t+k}\right]\left[A_t + y_t - c_t\right] + \left[\prod_{k=1}^{j-1} R_{t+k}\right]\left[y_{t+1} - c_{t+1}\right] + \left[\prod_{k=2}^{j-1} R_{t+k}\right]\left[y_{t+2} - c_{t+2}\right]$$

$$+ \dots + \left[\prod_{k=j-2}^{j-1} R_{t+k}\right]\left[y_{t+j-2} - c_{t+j-2}\right] + R_{t+j-1}\left[y_{t+j-1} - c_{t+j-1}\right] \qquad (e)$$

$$Let \quad j \to \infty, \qquad \lim_{j\to\infty} A_{t+j} = 0, \quad and \quad eq.(e) \ becomes$$

$$0 = \left[\prod_{k=0}^{\infty} R_{t+k}\right]\left[A_t + y_t - c_t\right] + \left[\prod_{k=1}^{\infty} R_{t+k}\right]\left[y_{t+1} - c_{t+1}\right]$$

$$+ \ldots . + \left[\prod_{k=2}^{\infty} R_{t+k}\right]\left[y_{t+j-2} - c_{t+j-2}\right] + R_{t+j-1}\left[y_{t+j-1} - c_{t+j-1}\right], \qquad (f)$$

$$\left[\prod_{k=0}^{\infty} R_{t+k}\right] c_t + \left[\prod_{k=1}^{\infty} R_{t+k}\right] c_{t+1} + \left[\prod_{k=2}^{\infty} R_{t+k}\right] c_{t+2} + \ldots ..$$

$$= \left[\prod_{k=0}^{\infty} R_{t+k}\right] y_t + \left[\prod_{k=1}^{\infty} R_{t+k}\right] y_{t+1} + \ldots .. \left[\prod_{k=0}^{\infty} R_{t+k}\right] A_t, \quad (g)$$

$$Then, \ multiplying \ both \ sides \ of \ \ eq.(g) \ by \left[\prod_{k=0}^{\infty} R_{t+k}\right]^{-1},$$

$$c_t + \left(R_t\right)^{-1} c_{t+1} + \left(R_t . R_{t+1}\right)^{-1} c_{t+2} + \ldots\ldots\ldots$$

$$= y_t + \left(R_t\right)^{-1} y_{t+1} + \left(R_t . R_{t+1}\right)^{-1} y_{t+2} + \ldots\ldots\ldots + A_t,$$

Hence,

$$c_t + \sum_{j=1}^{\infty}\left(\prod_{k=0}^{j-1} R_{t+k}^{-1}\right) c_{t+j} = y_t + \sum_{j=1}^{\infty}\left(\prod_{k=0}^{j-1} R_{t+k}^{-1}\right) y_{t+j} + A_t .$$

Saving under certainty

- Bellman's equation

$$V(A_t, y_t, R_{t-1}) = \max_{u_t}\left[U(A_t + y_t - u_t) + \beta V(R_t u_t, y_{t+1}, R_t)\right]$$

$$\textit{where} \qquad u_t = R_t^{-1} A_{t+1},$$

- Note that $c_t = A_t + y_t - u_t$, or $u_t = A_t + y_t - c_t$. Hence by recalling (9), which is the same marginal condition as (1.4b), and suppose that the optimal choice $u_t = h(A_t, y_t, R_{t-1})$ then we have

$$\frac{\partial V(A_t, y_t, R_{t-1})}{\partial A_t} = \frac{\partial U(c_t)}{\partial c_t}\frac{\partial\left(A_t + y_t - h(A_t, y_t, R_{t-1})\right)}{\partial A_t}$$

$$+\beta\frac{\partial V\left(R_t h(A_t, y_t, R_{t-1}), y_{t+1}, R_t\right)}{\partial R_t h(A_t, y_t, R_{t-1})}.R_t.\frac{\partial h(A_t, y_t, R_{t-1})}{\partial A_t},$$

Saving under certainty

$$\frac{\partial V\left(A_t, y_t, R_{t-1}\right)}{\partial A_t} = \frac{\partial U\left(c_t\right)}{\partial c_t}\left[1 - \frac{\partial h\left(A_t, y_t, R_{t-1}\right)}{\partial A_t}\right]$$

$$+\beta \frac{\partial V\left(R_t h\left(A_t, y_t, R_{t-1}\right), y_{t+1}, R_t\right)}{\partial R_t h\left(A_t, y_t, R_{t-1}\right)} R_t \frac{\partial h\left(A_t, y_t, R_{t-1}\right)}{\partial A_t},$$

$$= U'(c_t) - \frac{\partial h\left(A_t, y_t, R_{t-1}\right)}{\partial A_t}\left[U'(c_t) - \beta V'\left(A_{t+1}, y_{t+1}, R_t\right) R_t\right],$$

$$= U'(c_t), \qquad (\because First-order\ condition\ w.r.t.\ u_t)$$

The above result also valid for $t+1$, *hence,*

$$\frac{\partial V\left(A_{t+1}, y_{t+1}, R_t\right)}{\partial A_{t+1}} = \frac{\partial U\left(c_{t+1}\right)}{\partial c_{t+1}} = U'(c_{t+1}), \qquad (11)$$

Next, *by differentiating Bellman equation w.r.t.* u_t, *one has*

$$\frac{\partial U\left(A_t + y_t - u_t\right)}{\partial\left(A_t + y_t - u_t\right)} \frac{\partial\left(A_t + y_t - u_t\right)}{\partial u_t} + \beta \frac{\partial V\left(u_t R_t, y_{t+1}, R_t\right)}{\partial u_t R_t} \frac{\partial u_t R_t}{\partial u_t} = 0,$$

$$-U'\left(c_t\right) + \beta V'\left(A_{t+1}, y_{t+1}, R_t\right) R_t = 0,$$

$$-U'\left(c_t\right) + \beta R_t U'\left(c_{t+1}\right) = 0, \qquad \left[\because Eq.\, 11\right] \qquad (12)$$

*Eq.*12 *is the Euler equation for* u_t,

$$If\ \ U\left(c_t\right) = \ln c_t,\ then\ from\ (12)$$

$$c_{t+j} = \beta^j \left[R_t . R_{t+1} \ldots . R_{t+j-1}\right] c_t$$

Substituing this int*o the left hand side of* $\left(10\right)$, *which is*

$$c_t + \sum_{j=1}^{\infty} \left(\prod_{k=0}^{j-1} R_{t+k}^{-1} \right) c_{t+j},\ then$$

$$\begin{aligned} c_t + \sum_{j=1}^{\infty} \left(\prod_{k=0}^{j-1} R_{t+k}^{-1} \right) c_{t+j} &= c_t + \sum_{j=1}^{\infty} \left[\left(\prod_{k=0}^{j-1} R_{t+k}^{-1} \right) \left(\beta^j \left[R_t . R_{t+1} \ldots . R_{t+j-1}\right] c_t \right) \right] \\ &= c_t \left[1 + \sum_{j=1}^{\infty} \left[\left(\prod_{k=0}^{j-1} R_{t+k}^{-1} \right) \left(\beta^j \left[R_t . R_{t+1} \ldots . R_{t+j-1}\right] \right) \right] \right] \\ &= c_t \left[1 + \left\{ \beta + \beta^2 + \ldots\ldots\ldots. \right\} \right] \\ &= c_t \left(1 - \beta\right)^{-1} \end{aligned}$$

Hence, (10) *and* (12) *imply that*

$$c_t = \left(1 - \beta\right) \left[y_t + \sum_{j=1}^{\infty} \left(\prod_{k=0}^{j-1} R_{t+k}^{-1} \right) y_{t+j} + A_t \right] \qquad \left(13\right)$$

Exercise

$$\max \quad \sum_{t=0}^{\infty} \beta^t U(c_t), \qquad 0 < \beta < 1$$

Subject to

$$A_{t+1} = R_t (A_t + y_t - c_t),$$

$$or, \quad c_t + \sum_{j=1}^{\infty} \left(\prod_{k=0}^{j-1} R_{t+k}^{-1} \right) c_{t+j} = y_t + \sum_{j=1}^{\infty} \left(\prod_{k=0}^{j-1} R_{t+k}^{-1} \right) y_{t+j} + A_t. \qquad (10)$$

A_0 *given,*

- Given that $U(c_t) = \frac{c_t^{1-\theta}}{1-\theta}, \qquad 0 < \theta < 1$
- and $R_t = R, \quad \forall t.$

Exercise

- The Euler equation is,

- $-c_t^{-\theta} + \beta R c_{t+1}^{-\theta} = 0,$
- Or $c_{t+j} = \beta^{j/\theta}\left[R_t . R_{t+1} \ldots . R_{t+j-1}\right]$
- From (10), one has life-time consumption expenditure as,
- $c_t + \sum_{j=1}^{\infty}\left[\prod_{k=0}^{j-1} R_{t+k}^{-1}\right]\left[\beta^{j/\theta}\left[R_t . R_{t+1} \ldots R_{t+j-1}\right]^{1/\theta}\right]$
- $= c_t\left[1 + \beta^{1/\theta} R_t^{\frac{1}{\theta}-1} + \beta^{2/\theta} R_{t+1}^{\frac{1}{\theta}-1} + \cdots \ldots .\right]$

Exercise

- $= c_t \left[1 + \beta^{1/\theta} R_t^{\frac{1}{\theta}-1} + \beta^{2/\theta} R_{t+1}^{\frac{1}{\theta}-1} + \cdots\right]$
- $= c_t \left[1 + \beta^{\frac{1}{\theta}} R^{\frac{1-\theta}{\theta}} [1 + \beta + \beta^2 + \beta^3 + \cdots]\right]$
- $= c_t \left[1 + \left[\frac{\beta^{\frac{1}{\theta}} R^{\frac{1-\theta}{\theta}}}{1-\beta}\right]\right]$
- $c_t \left[\frac{1-\beta^{\frac{1}{\theta}}\left[\beta^{\frac{\theta-1}{\theta}} + R^{\frac{1-\theta}{\theta}}\right]}{1-\beta}\right]$

Optimal Growth under certainty

$$\max \quad \sum_{t=0}^{\infty} \beta^t U\left(c_t\right), \qquad 0<\beta<1$$

$$s.t. \qquad c_t + k_{t+1} = f\left(k_t\right), \qquad k_0 > 0 \;\; given, \quad c_t \geq 0$$

$$U'\left(0\right) = +\infty, \;\; U' > 0, \;\; U'' < 0,$$

$$f'(0) = \infty, \;\; f'(\infty) = 0, \;\; f' > 0, f'' < 0.$$

$$(\therefore \; one \; may \; write \;\; k_{t+1} = f\left(k_t\right) - c_t = g\left(c_t, k_t\right))$$

Optimal Growth under certainty

$$Bellman's\ equation,$$

$$V\left(k_t\right) = \max_{k_{t+1}} \left\{ U\left[f(k_t) - k_{t+1}\right] + \beta V\left(k_{t+1}\right) \right\}$$

$$Euler\ equation\ \ for\ k_{t+1}\ ,$$

$$\frac{\partial U\left[f(k_t) - k_{t+1}\right]}{\partial\left[f(k_t) - k_{t+1}\right]} \frac{d\left[f(k_t) - k_{t+1}\right]}{d\left[k_{t+1}\right]} + \beta \frac{\partial V\left(k_{t+1}\right)}{\partial k_{t+1}} = 0,$$

$$\frac{\partial U\left(c_t\right)}{\partial c_t}\left(-1\right) + \beta \frac{\partial V\left(k_{t+1}\right)}{\partial k_{t+1}} = 0,$$

$$\frac{\partial U\left(c_t\right)}{\partial c_t} = \beta \frac{\partial V\left(k_{t+1}\right)}{\partial k_{t+1}}, \qquad (14)$$

Optimal Growth

From (9), (*The marginal condition*), *and the optimal choice* $k_{t+1} = h(k_t)$

$$V'(k_t) = \frac{\partial U\left[f(k_t) - h(k_t)\right]}{\partial\left[f(k_t) - h(k_t)\right]}\left[\frac{df(k_t)}{dk_t} - \frac{dh(k_t)}{dk_t}\right] + \beta\frac{\partial V(h(k_t))}{\partial h(k_t)}\frac{dh(k_t)}{dk_t}$$

$$= \frac{\partial U(c_t)}{\partial c_t}\left[f'(k_t)\right] - \frac{\partial h(k_t)}{k_t}\left[\frac{\partial U(c_t)}{\partial c_t} - \beta V'(k_{t+1})\right]$$

$$= \frac{\partial U(c_t)}{\partial c_t}\left[f'(k_t)\right], \qquad (\because\ eq.(14))$$

The above result is also valid at $t+1$, *hence*

$$V'(k_{t+1}) = \frac{\partial U(c_{t+1})}{\partial c_{t+1}}\left[f'(k_{t+1})\right], \tag{15}$$

Hence, substitute (15) into (14), *one has*

$$-U'(c_t) + \beta U'(c_{t+1})f'(k_{t+1}) = 0 \tag{16}$$

Optimal Growth

$$\text{From } (16), \qquad f'(k_{t+1}) = \frac{U'(c_t)}{\beta U'(c_{t+1})},$$

$$= \frac{U'\left(f(k_t) - k_{t+1}\right)}{\beta U'\left(f(k_{t+1}) - k_{t+2}\right)}.$$

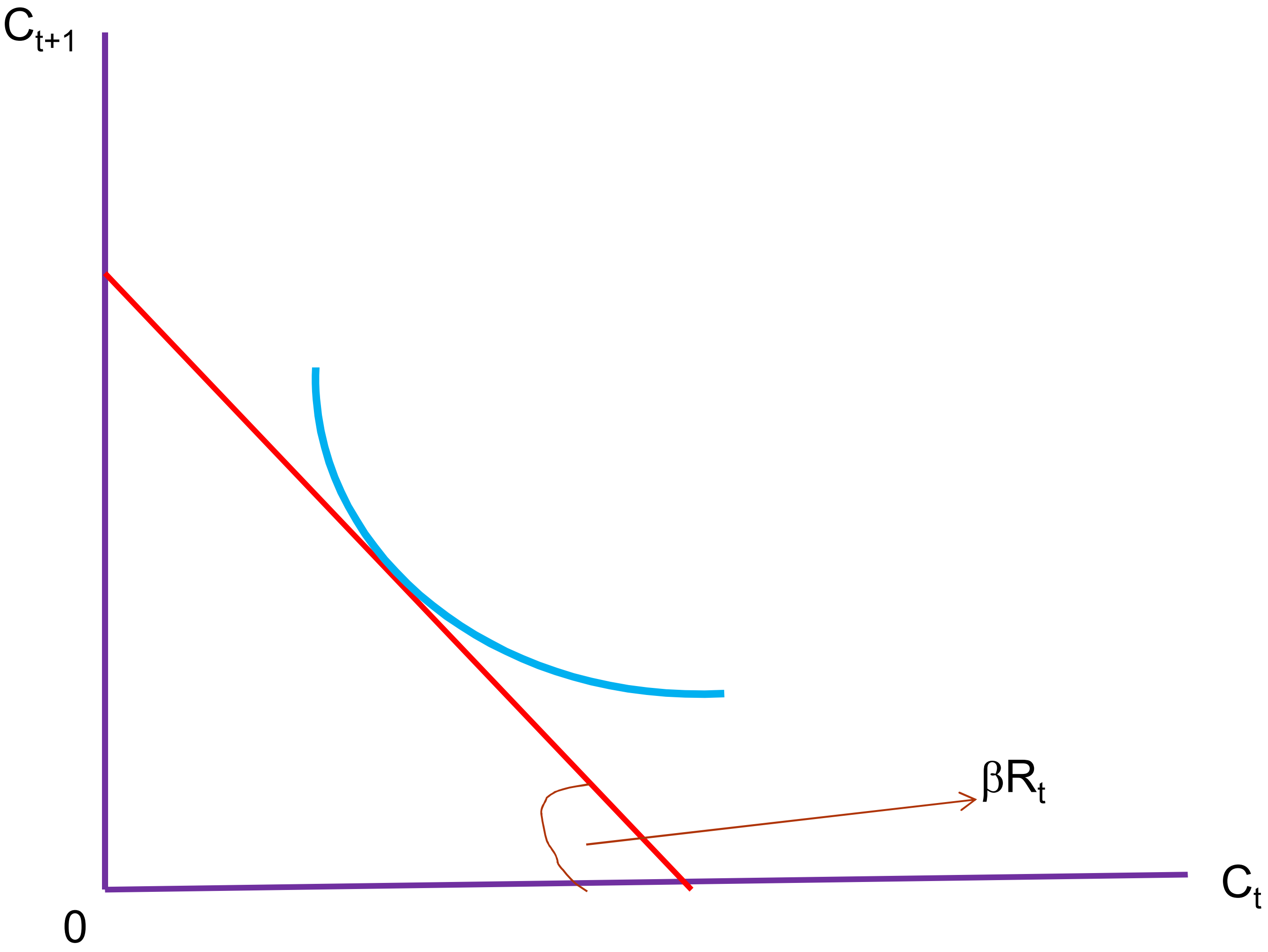
$\therefore$ *Optimal policy function* $k_{t+2} = h(k_{t+1})$ *must be a non decreasing function of* k_{t+1}. *This is becasue*

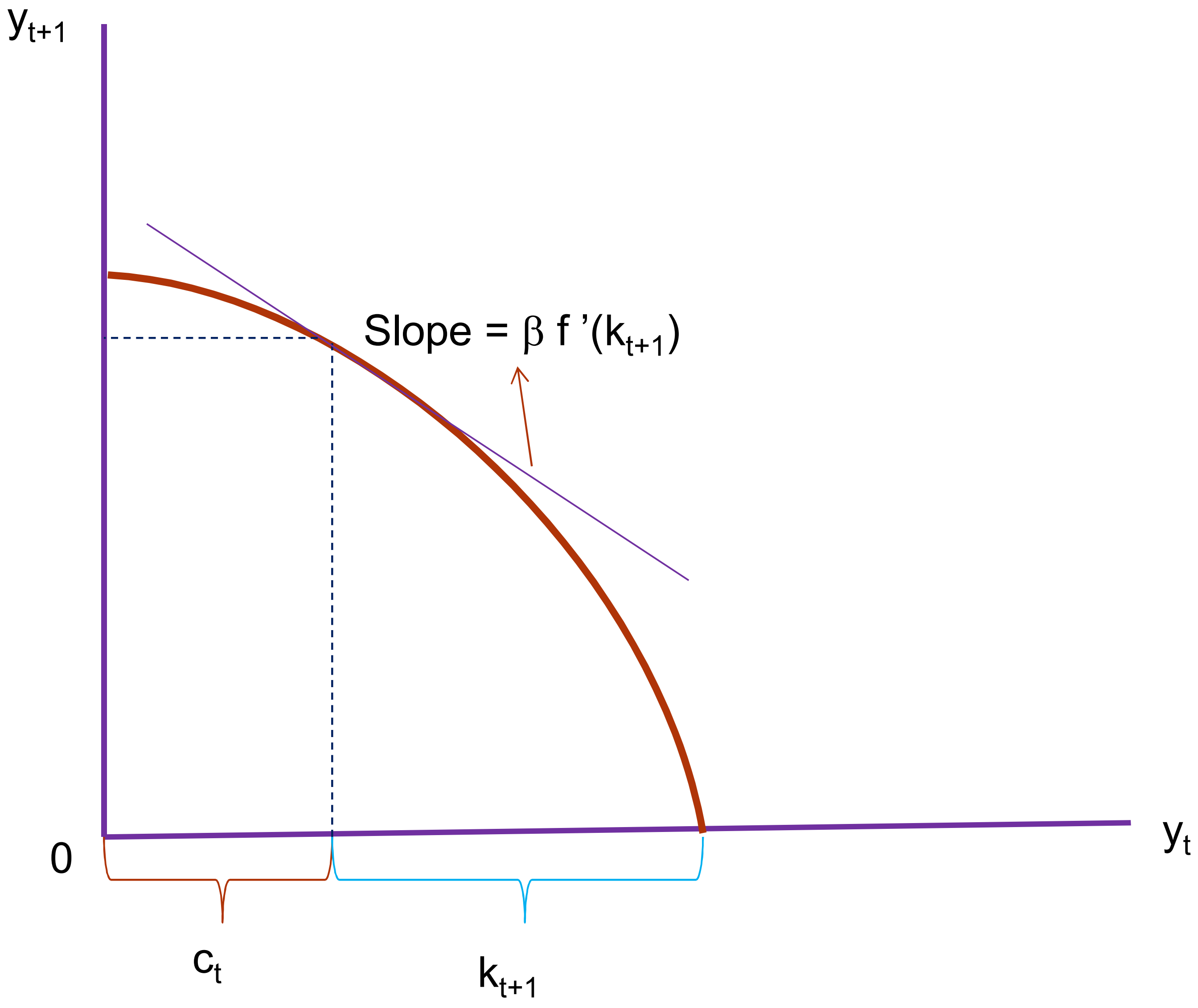
$$f''(k_{t+1}) =$$

$$\frac{\left[\beta U'(c_{t+1})\right]\left[U''(c_t).(-1)\right] - \left[U'(c_t)\right]\left[\beta U''(c_{t+1}).\left(f'(k_{t+1}) - \frac{\partial h(k_{t+1})}{\partial k_{t+1}}\right)\right]}{\left[\beta U'(c_{t+1})\right]^2} \leq 0$$

$\Rightarrow$ *Suppose that* $\frac{\partial h(k_{t+1})}{\partial k_{t+1}} < 0$, *then the RHS of the above equation will be greater than zero, which is contradicted to* $f''(k_{t+1}) \leq 0$. *Hence, it must be that*

$$\frac{\partial h(k_{t+1})}{\partial k_{t+1}} \geq 0.$$


C_{t+1}
βR_t
C_t
0


y_{t+1}
Slope = $\beta f'(k_{t+1})$
0
y_t
c_t
k_{t+1}