EE320 (2/2013)

INTRODUCTORY MATHEMATICAL ECONOMICS

OPTIMIZATION UNDER EQUALITY CONSTRAINTS

Topics

- Effects of a constraint
- Finding the stationary (optimal) values
  - Elimination method
  - Lagrange multiplier
  - *n*-variable and multi-constraint cases
- Second-order conditions for constrained optimization
- Economic applications
- Extensions: n-variable and multiconstraint cases

Effects of a Constraint

- In the previous topic, we studied optimization problems, where the choice variables can be chosen freely.
- Example: A profit-maximizing firm's objective is:

$$\pi = PQ - [C(Q_1) + C(Q_2)]$$

where $Q = Q_1 + Q_2$.

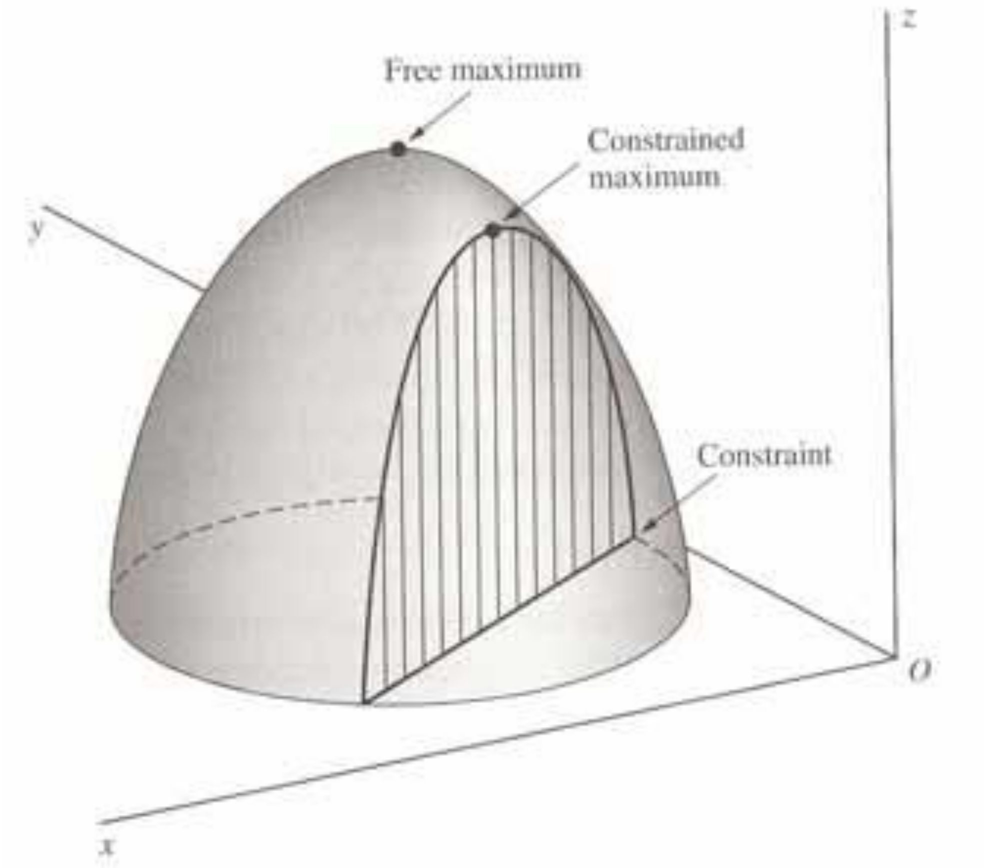
  - Q_1 and Q_2 can be chosen freely in order to maximize profit.

    ➔ Here, Q_1^* and Q_2^* are *free optimal values*.

  - Question: If there is a *production constraint*, e.g. $Q_1 + Q_2 = 900$, then how would the firm change its behavior?
  - ➔ Q_1^{**} and Q_2^{**} will now become the *constrained optimal values*.

Effects of a Constraint

- **Figure:** "Free extremum" vs. "Constrained extremum"



FINDING THE STATIONARY VALUES

Two Choice Variables with One Equality Constraint

- Use *elimination* (or *substitution*) method

Example: $\max\limits_{x_1, x_2} U = x_1 x_2 + 2x_1$

subject to $4x_1 + 2x_2 = 60$

Two Choice Variables with One Constraint

- Use *Lagrange-multiplier* method

Consider the following maximization problem.

$$\max_{x,y} f(x,y)$$

subject to $g(x,y) = c$.

Define the Lagrangian function by

$$L(x,y,\lambda) = f(x,y) + \lambda[c - g(x,y)]$$

F.O.N.C.

$$L_x(x,y) = f_x(x,y) - \lambda g_x(x,y) = 0$$

$$L_y(x,y) = f_y(x,y) - \lambda g_y(x,y) = 0$$

$$L_\lambda(x,y) = c - g(x,y) = 0$$

Example 1: 2 choice variables & 1 constraint

- Find the optimal values for the following function.

$$U = x_1 x_2 + 2x_1$$

subject to $4x_1 + 2x_2 = 60$

Step 1- Set up the Lagrangian function:

$$L(x_1, x_2, \lambda) = x_1 x_2 + 2x_1 + \lambda[60 - 4x_1 - 2x_2]$$

Step 2 - Find FONC:

Example 2: 2 choice variables & 1 constraint

- Find the optimal values for the following function:

$$f\left(x_1, x_2\right) = x_1^2 + x_2^2$$

subject to $g(x_1, x_2) = x_1 + 4x_2 = 2$

Step 1- Set up the Lagrangian function:

$$L(x_1, x_2, \lambda) = x_1^2 + x_2^2 + \lambda[2 - x_1 - 4x_2]$$

Step 2 - Find FONC:

Interpretation of Lagrange Multiplier

- Consider the problem

$$\max_{x,y}(\min)\ f(x,y) \text{ subject to } g(x,y)=c$$

  - Solutions to this problem: *x*(c), y*(c),* and *f*(c).*
  - *f*(c)* is called optimal value function.

- We can show that $\dfrac{df^*(c)}{dc}=\lambda(c)$

  - The Lagrange multiplier λ is the rate at which the optimal value of the objective function changes with respect to changes in the constraint constant, c.

Interpretation of Lagrange Multiplier

- In economic applications, *c* denotes the available stock of some resources, *f(x, y)* denotes utility or profit.
- $\lambda(c)dc$ is the change in utility or profit that can be obtained from *dc* units more of the resource.
- Thus, λ is called a "shadow price" of the resource.
- Examples:
  - For cost minimization problem, $\lambda = \frac{dC}{dQ_0} = MC$
  - For utility maximization problem, $\lambda = \frac{dU}{dY_0}$ = MU of income
  - For profit maximization problem, $\lambda = \frac{d\pi}{dQ_0}$
  - For output maximization problem, $\lambda = \frac{dQ}{dC_0}$

SECOND-ORDER CONDITIONS FOR CONSTRAINED OPTIMIZATION

Second-Order Total Differential

- Recall the FONC for $L(x,y,\lambda)=f(x,y)+\lambda[c-g(x,y)]$

$$L_x(x,y)=f_x(x,y)-\lambda g_x(x,y)=0$$

$$L_y(x,y)=f_y(x,y)-\lambda g_y(x,y)=0$$

- By partially differentiating the FONC, the second derivatives are:

$$L_{xx}$$

$$L_{xy}$$

$$L_{yy}$$

- Thus, d^2z can be rewritten as:

$$d^2z = L_{xx}dx^2+L_{xy}dxdy$$
$$+L_{yx}dydx+L_{yy}dy^2$$

Second-Order Conditions

- The second-order sufficient conditions are:
  - For *maximum* of z: d^2z *is negative definite*, subject to $dg = 0$
  - For *minimum* of z: d^2z *is positive definite*, subject to $dg = 0$

  where $dg = g_x dx + g_y dy = 0$.

- Define the determinant of the *Bordered Hessian matrix* as:

$$|\overline{H}| = \begin{vmatrix} 0 & g_x & g_y \\ g_x & L_{xx} & L_{xy} \\ g_y & L_{yx} & L_{yy} \end{vmatrix}$$

where $g_x = \partial g/\partial x$; $g_y = \partial g/\partial y$;

and $L_{xx}, L_{xy}, L_{yx}, L_{yy}$ are the second-order derivatives of $L(x, y, \lambda)$.

Second-Order Conditions

- The sign definiteness of d^2z can be determined from the following criterion:*
  - d^2z is negative definite subject to $g_x dx + g_y dy = 0$ iff $|\bar{H}| > 0$.
  - d^2z is positive definite subject to $g_x dx + g_y dy = 0$ iff $|\bar{H}| < 0$.

- Summary:

For a stationary value of $L(x, y) = f(x, y) + \lambda[c - g(x, y)]$

The second-order sufficient conditions are:

For *maximum* of z, the *bordered Hessian is positive* ($|\bar{H}| > 0$);

For *minimum* of z, the *bordered Hessian is negative* ($|\bar{H}| < 0$).

*See additional notes on the determinantal tests for the sign definiteness of d^2z.

Example

- Determine whether the extremum of the objective function

$U = x_1 x_2 + 2x_1$ subject to $4x_1 + 2x_2 = 60$

gives a minimum or a maximum.

Example

- Determine whether the extremum of the objective function

$$f(x_1, x_2) = x_1^2 + x_2^2 \quad \text{subject to} \quad g(x_1, x_2) = x_1 + 4x_2 = 2$$

gives a minimum or a maximum.

ECONOMIC APPLICATIONS OF CONSTRAINED OPTIMIZATION

Utility Maximization Problem

- Objective function: $\max_{Q_1,Q_2} U = U(Q_1,Q_2)$
- Subject to: $Y_0 = p_1Q_1 + p_2Q_2$

➢ Lagrangian function:

$$L(Q_1,Q_2,\lambda) = U(Q_1,Q_2) + \lambda[Y_0 - p_1Q_1 - p_2Q_2]$$

➢F.O.N.C.

$$L_1 = U_1(Q_1,Q_2) - \lambda p_1 = 0$$

$$L_2 = U_2(Q_1,Q_2) - \lambda p_2 = 0$$

$$L_\lambda = Y_0 - p_1Q_1 - p_2Q_2 = 0$$

➔ (Q_1^*,Q_2^*) is such that $\frac{U_1(Q_1,Q_2)}{U_2(Q_1,Q_2)} = \frac{p_2}{p_1}$ and $Y_0 = p_1Q_1 + p_2Q_2$.

Utility Maximization Problem

- Second-order sufficient condition

$$\left|\overline{H}\right| = \begin{vmatrix} 0 & g_x & g_y \\ g_x & L_{xx} & L_{xy} \\ g_y & L_{yx} & L_{yy} \end{vmatrix} = \begin{vmatrix} 0 & p_1 & p_2 \\ p_1 & U_{11} & U_{12} \\ p_2 & U_{21} & U_{22} \end{vmatrix} = -p_1^2 U_{22} + 2p_1 p_2 U_{12} - p_2^2 U_{11} > 0$$

Utility Maximization Problem

- Example: Suppose that $U = 50A + 200B - 0.5A^2 - 2.5B^2$. Find A and B that maximize the utility given that $P_A = 10$, $P_B = 5$, and $Y = 490$.

Expenditure Minimization

- Objective function: $\underset{Q_1,Q_2}{Min} E = p_1Q_1 + p_2Q_2$
- Subject to: $\overline{U} = U(Q_1,Q_2)$

➢ Lagrangian function:

$$L(Q_1,Q_2,\lambda) = p_1Q_1 + p_2Q_2 + \lambda[\overline{U} - U(Q_1,Q_2)]$$

➢F.O.N.C.

$$\left.\begin{aligned} L_1 &= p_1 - \lambda U_1(Q_1,Q_2) = 0 \\ L_2 &= p_2 - \lambda U_2(Q_1,Q_2) = 0 \\ L_\lambda &= \overline{U} - U(Q_1,Q_2) = 0 \end{aligned}\right\}$$

$$\boxed{\begin{array}{l} (Q_1^*,Q_2^*) \text{ where} \\ \dfrac{U_1(Q_1,Q_2)}{U_2(Q_1,Q_2)} = \dfrac{p_1}{p_2} \text{ and } \overline{U} = U(Q_1,Q_2) \end{array}}$$

➢S.O.S.C. $|\overline{H}| < 0$

Expenditure Minimization

- Example: Suppose that $U = 50A + 200B – 0.5A^2 – 2.5B^2$, and the consumer would like to have a utility fixed at U_0. Find A and B that minimize the expenditure given that $P_A = 10$ and $P_B = 5$.

Profit Maximization Problem

- Objective function: $\max\limits_{Q_1,Q_2} \pi = p_1Q_1 + p_2Q_2 - c(Q_1,Q_2)$
- Subject to: $\overline{Q} = Q_1 + Q_2$

➢ Lagrangian function:

$$L(Q_1,Q_2,\lambda) = p_1Q_1 + p_2Q_2 - c(Q_1,Q_2) + \lambda[\overline{Q} - Q_1 - Q_2]$$

➢ F.O.N.C.

$$\left.\begin{aligned} L_1 &= p_1 - c_1(Q_1,Q_2) - \lambda = 0 \\ L_2 &= p_2 - c_2(Q_1,Q_2) - \lambda = 0 \\ L_\lambda &= \overline{Q} - Q_1 - Q_2 = 0 \end{aligned}\right\}$$

(Q_1^*, Q_2^*) where

$p_1 - c_1 = p_2 - c_2$ and $\overline{Q} = Q_1 + Q_2$

➢ S.O.S.C. $|\overline{H}| > 0$

Profit Maximization Problem

- Example: Suppose that P_1 = 80, P_2 = 100, $TC = 100 + 0.1Q_1^2 + 0.2Q_2^2$

And $Q_1 + Q_2 = 325$. Find Q_1 and Q_2 that maximize the profit.

Output Maximization Problem

- Objective function: $\max\limits_{K,L} Q = Q(K,L)$
- Subject to: $\overline{C} = wL + rK$

➢ Lagrangian function:

$$L(K,L,\lambda) = Q(K,L) + \lambda[\overline{C} - wL - rK]$$

➢ F.O.N.C.

$$\left.\begin{aligned} L_K &= Q_K(K,L) - \lambda r = 0 \\ L_L &= Q_L(K,L) - \lambda w = 0 \\ L_\lambda &= \overline{C} - wL - rK = 0 \end{aligned}\right\}$$

(K^*,L^*) where

$\dfrac{Q_K(K,L)}{Q_L(K,L)} = \dfrac{r}{w}$ and $\overline{C} = wL + rK$

➢S.O.S.C. $|\overline{H}| > 0$

Output Maximization Problem

- Example: Suppose $Q = KL$, w = 6, r = 10, C_0 = 60. Find K and L that maximizes output.

Cost Minimization

- Objective function: $\underset{K,L}{Min}\, C = wL + rK$
- Subject to: $\overline{Q} = f(K,L)$

➢ Lagrangian function:

$$L(K,L,\lambda) = wL + rK + \lambda[\overline{Q} - Q(K,L)]$$

➢ F.O.N.C.

$$\left.\begin{aligned} L_K &= r - \lambda f_K(K,L) = 0 \\ L_L &= w - \lambda f_L(K,L) = 0 \\ L_\lambda &= \overline{Q} - f(K,L) = 0 \end{aligned}\right\}$$

(K^*, L^*) where

$\dfrac{f_K(K,L)}{f_L(K,L)} = \dfrac{r}{w}$ and $\overline{Q} = f(K,L)$

➢ S.O.S.C. $|\overline{H}| < 0$

Cost Minimization

- Graph: Expansion path

➔ Expansion path describes the least cost combination of K* and L* required to produce varying levels of quantities.

Cost Minimization

- Example: Suppose $Q = KL$. A firm will produce 15 units of the good. Find K and L that minimize total cost given that w = 6, r =10.

EXTENSION: N-CHOICE VARIABLE AND MULTI-CONSTRAINT CASES

Optimization with Equality Constraints: *n*-Variable and One Equality Constraint

- The objective function

$$z = f(x_1, x_2, \ldots, x_n)$$

Subject to the constraint $g(x_1, x_2, \ldots, x_n) = c$

The Lagrangian function:

$$L(x_1, \ldots, x_n, \lambda) = f(x_1, \ldots, x_n) + \lambda[c - g(x_1, \ldots, x_n)]$$

FONC:

$$L_1 = f_1(x_1, \ldots, x_n) - \lambda g_1(x_1, \ldots, x_n) = 0$$

$$\vdots$$

$$L_n = f_n(x_1, \ldots, x_n) - \lambda g_n(x_1, \ldots, x_n) = 0$$

$$L_\lambda(x_1, \ldots, x_n) = c - g(x_1, \ldots, x_n) = 0$$

Second-Order Conditions: *n*-Variable and One Equality Constraint

- The objective function $z = f(x_1, x_2, \ldots, x_n)$

  Subject to $g(x_1, x_2, \ldots, x_n) = c$

- **Bordered Hessian:**

$$|\bar{H}| = \begin{vmatrix} 0 & g_1 & g_2 & \cdots & g_n \\ g_1 & L_{11} & L_{12} & \cdots & L_{1n} \\ g_2 & L_{21} & L_{22} & \cdots & L_{2n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ g_n & L_{n1} & L_{n2} & \cdots & L_{nn} \end{vmatrix}$$

- Bordered leading principal minors can be defined as:

$$|\bar{H}_2| = \begin{vmatrix} 0 & g_1 & g_2 \\ g_1 & L_{11} & L_{12} \\ g_2 & L_{21} & L_{22} \end{vmatrix}; |\bar{H}_3| = \begin{vmatrix} 0 & g_1 & g_2 & g_3 \\ g_1 & L_{11} & L_{12} & L_{13} \\ g_2 & L_{21} & L_{22} & L_{23} \\ g_3 & L_{31} & L_{32} & L_{33} \end{vmatrix}; \ldots; |\bar{H}_n| = |\bar{H}|$$

(Note: The subscript refers to the size of the bordered matrix.)

Second-Order Conditions:
n-Variable and One Equality Constraint

- The conditions for positive and negative definiteness of d^2z are:

d^2z is $\begin{Bmatrix} positive\_definite \\ negative\_definite \end{Bmatrix}$ subject to $dg = 0$ iff $\begin{cases} |\bar{H}_2|, |\bar{H}_3|, \ldots, |\bar{H}_n| < 0 \\ |\bar{H}_2| > 0; |\bar{H}_3| < 0; |\bar{H}_4| > 0; etc. \end{cases}$

Condition	Maximum	Minimum
First-order necessary condition	$L_\lambda = L_1 = L_2 = \ldots = L_n = 0$	$L_\lambda = L_1 = L_2 = \ldots = L_n = 0$
Second-order necessary condition*	$|\bar{H}_2| > 0; |\bar{H}_3| < 0;$ $|\bar{H}_4| > 0; \ldots; (-1)^n |\bar{H}_n| > 0$	$|\bar{H}_2|, |\bar{H}_3|, \ldots, |\bar{H}_n| < 0$

Cost Minimization

- Objective function: $\underset{K,L,T}{Min} C = wL + iK + rT$
- Subject to: $Q_0 = f(K, L, T)$

➢ Lagrangian function:

Optimization with Equality Constraints: *n*-Variable and Two Equality Constraints

- The objective function

$$z = f(x_1, x_2, \ldots, x_n)$$

Subject to the constraints

$$g(x_1, x_2, \ldots, x_n) = c$$
$$h(x_1, x_2, \ldots, x_n) = d$$

- **The Lagrangian function:**

$$L = f(x_1, x_2, \ldots, x_n) + \lambda_1 [c - g(x_1, x_2, \ldots, x_n)] + \lambda_2 [d - h(x_1, x_2, \ldots, x_n)]$$

- **FONC:**

$$L_i = f_i(x_1, \ldots, x_n) - \lambda_1 g_i(x_1, \ldots, x_n) - \lambda_2 h_i(x_1, \ldots, x_n) = 0$$ for *i = 1, …, n*

$$L_{\lambda_1}(x_1, \ldots, x_n) = c - g(x_1, \ldots, x_n) = 0$$

$$L_{\lambda_2}(x_1, \ldots, x_n) = d - h(x_1, \ldots, x_n) = 0$$

Optimization with Equality Constraints:

n-Variable and *k* Equality Constraints

- The objective function

$$z = f(x_1, x_2, \ldots, x_n)$$

Subject to the constraints

$$g_1(x_1, x_2, \ldots, x_n) = c_1$$
$$g_2(x_1, x_2, \ldots, x_n) = c_2$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$g_k(x_1, x_2, \ldots, x_n) = c_k$$

➢ **The Lagrangian function:**

$$L = f(x_1, x_2, \ldots, x_n) + \lambda_1\left[c_1 - g^1(x_1, x_2, \ldots, x_n)\right] + \lambda_2\left[c_2 - g^2(x_1, x_2, \ldots, x_n)\right]$$
$$+ \ldots + \lambda_k\left[c_k - g^k(x_1, x_2, \ldots, x_n)\right]$$

Optimization with Equality Constraints:

n-Variable and *k* Equality Constraints

- **FONC:**

$$\frac{\partial L}{\partial x_1} = f_1(x_1,\dots,x_n) - \lambda_1 g_1^1(x_1,\dots,x_n) - \dots - \lambda_k g_1^k(x_1,\dots,x_n) = 0$$

$$\frac{\partial L}{\partial x_2} = f_2(x_1,\dots,x_n) - \lambda_1 g_2^1(x_1,\dots,x_n) - \dots - \lambda_k g_2^k(x_1,\dots,x_n) = 0$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\frac{\partial L}{\partial x_n} = f_n(x_1,\dots,x_n) - \lambda_1 g_n^1(x_1,\dots,x_n) - \dots - \lambda_k g_n^k(x_1,\dots,x_n) = 0$$

$$\frac{\partial L}{\partial \lambda_1} = c_1 - g^1(x_1,\dots,x_n) = 0$$

$$\frac{\partial L}{\partial \lambda_2} = c_2 - g^2(x_1,\dots,x_n) = 0$$

$$\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\frac{\partial L}{\partial \lambda_k} = c_k - g^k(x_1,\dots,x_n) = 0$$

Second-Order Conditions: *n*-Variable and *k* Equality Constraints

The Lagrangian condition is:

$$L = f(x_1, x_2, \dots, x_n) + \sum_{j=1}^{k} \lambda_j \left[c_j - g^j(x_1, x_2, \dots, x_n) \right]$$

- Bordered Hessian:

$$|\overline{H}| \equiv \left| \begin{array}{cccc|cccc} 0 & 0 & \cdots & 0 & g_1^1 & g_2^1 & \cdots & g_n^1 \\ 0 & 0 & \cdots & 0 & g_1^2 & g_2^2 & \cdots & g_n^2 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & g_1^k & g_2^k & \cdots & g_n^k \\ \hline g_1^1 & g_1^2 & \cdots & g_1^k & L_{11} & L_{12} & \cdots & L_{1n} \\ g_2^1 & g_2^2 & \cdots & g_2^k & L_{21} & L_{22} & \cdots & L_{1n} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ g_n^1 & g_n^2 & \cdots & g_n^k & L_{n1} & L_{n2} & \cdots & L_{nn} \end{array} \right|$$

Second-Order Conditions: *n*-Variable and *k* Equality Constraints

- The second-order sufficient conditions are stated in terms of the signs of the following *(n-k)* bordered leading principal minors:

$$|\bar{H}_{k+1}|, |\bar{H}_{k+2}|, \ldots, |\bar{H}_n| \qquad \text{where } |\bar{H}_n| = |\bar{H}|$$

- SOSC:

1) For a *maximum of* z, the bordered leading principal minors alternate in sign, and the sign of $|\bar{H}_{k+1}| = (-1)^{k+1}$.
2) For a *minimum of* z, all the bordered leading principal minors take the same sign, namely that of $(-1)^k$.

Example

- The objective function

$$z = f(x_1, x_2, x_3)$$

Subject to $g(x_1, x_2, x_3) = c$

$$h(x_1, x_2, x_3) = d$$

➢The Lagrangian function:

$$L = f(x_1, x_2, x_3) + \lambda_1[c - g(x_1, x_2, x_3)] + \lambda_2[d - h(x_1, x_2, x_3)]$$

➢FONC: $L_1 = L_2 = L_3 = L_{\lambda_1} = L_{\lambda_2} = 0$

➢SOSC: 1) Max: $|H_3| < 0$

2) Min: $|H_3| > 0$