Pannatat Pongthiengtham
6204640731

6.5 Exercises (HW : Q1-5)

1. Two individuals agree at date 0 to a forward contract that matures at date 2. The contract is written on an underlying asset that pays a dividend at date 1 equal to D_1. Let f_2 be the date 2 random payoff (profit) to the individual who is the long party in the forward contract. Also let m_{0i} be the stochastic discount factor over the period from dates 0 to i where $i = 1, 2$, and let $E_0[\cdot]$ be the expectations operator at date 0. What is the value of $E_0[m_{02} f_2]$? Explain your answer.

P_i = price of asset at date i

- $P_0 = E_0[m_{01} D_1] + E_0[m_{02} P_2]$

  $P_0 = D_0 + E_0[m_{02} P_2]$; D_0 = present value of dividends

  $E_0[m_{02} P_2] = P_0 - D_0$ — (1)

- Long party
  $\hookrightarrow f_2 = P_2 - F_{02}$; F_{02} = forward price

  $E[m_{02} F_{02}] = E[m_{02}] F_{02}$; $E[m] = \frac{1}{R_f}$

  $E[m_{02} F_{02}] = R_f^{-2} F_{02}$ — (2)

- $E[m_{02} f_2] = E_0[m_{02}(P_2 - F_{02})]$

  $= E_0[m_{02} P_2] - E_0[m_{02} F_{02}]$

  $= P_0 - D_0 - R_f^{-2} F_{02}$

The absence of arbitrage

$E[m_{02} f_2] = 0$ when $F_{02} = R_f^{2}(P_0 - D_0)$

2. Assume that there is an economy populated by infinitely-lived representative individuals who maximize the lifetime utility function

$$E_0\left[\sum_{t=0}^{\infty} -\delta^t e^{-ac_t}\right]$$

where c_t is consumption at date t and $a > 0$, $0 < \delta < 1$. The economy is a Lucas (1978) endowment economy having multiple risky assets paying date t dividends that total d_t per capita. Write down an expression for the equilibrium per capita price of the market portfolio in terms of the assets' future dividends.

From Lucas model

$$P_0 = E_0\left[\sum_{t=1}^{\infty} \frac{U_c(C_t, t)}{U_c(C_0, 0)} d_0\right]$$

$U(C_t, t) = -\delta^t e^{-ac_t}$

$U_c(C_t, t) = a\delta^t e^{-ac_t}$

$U_c(C_0, 0) = a e^{-ac_0}$

$$\therefore P_0 = E_0\left[\sum_{t=1}^{\infty} \frac{\cancel{a}\delta^t e^{-ad_t}}{\cancel{a} e^{-ad_0}} d_t\right] \quad ; c_t = d_t$$

$$= E_0\left[\sum_{t=1}^{\infty} \delta^t e^{-a(d_t - d_0)} d_t\right]$$

3. For the Lucas model with labor income, show that assumptions (6.25) and (6.26) lead to the pricing relationship (6.27) and (6.28).

$$P_t = E_t\left[\sum_{j=1}^{\infty} \delta^j \left[\frac{C^*_{t+j}}{C^*_t}\right]^{\gamma-1} d_{t+1}\right]$$

$$\ln\left(\frac{C_{t+1}}{C_t}\right) = \mu_c + \sigma_c \eta_{t+1}$$

$$\ln\left(\frac{C_{t+2}}{C_t}\right) = \ln\left(\frac{C_{t+2}}{C_{t+1}}\right) + \ln\left(\frac{C_{t+1}}{C_t}\right)$$

$$= 2\mu_c + \sigma_c\eta_{t+1} + \sigma_c \eta_{t+2}$$

$$= 2\mu_c + \sum_{i=1}^{2}\sigma_c \eta_{t+i}$$

$$\therefore \ln\left(\frac{C_{t+j}}{C_t}\right) = j\mu_c + \sigma_c\sum_{i=1}^{j}\eta_{t+i}$$

or

$$\ln\left(\frac{d_{t+j}}{d_t}\right) = j\mu_d + \sigma_d \sum_{i=1}^{j}\varepsilon_{t+i}$$

$$\frac{C_{t+j}}{C_t} = e^{j\mu_c + \sigma_c\sum_{i=1}^{j}\eta_{t+i}}$$

$$\frac{P_t}{d_t} = E_t\left[\sum_{j=1}^{\infty}\delta^j\left[\frac{C^*_{t+j}}{C^*_t}\right]^{\gamma-1}\frac{d_{t+1}}{d_t}\right]$$

$$\frac{P_t}{d_t} = E_t\left[\sum_{j=1}^{\infty}\delta^j e^{(\gamma-1)\left[j\mu_c + \sigma_c\sum_{i=1}^{j}\eta_{t+i}\right] + \left[j\mu_d + \sigma_d\sum_{i=1}^{j} \varepsilon_{t+i}\right]}\right]$$

take expectation inside

$$\frac{P_t}{d_t} = \sum_{j=1}^{\infty}\delta^j e^{(\gamma-1)j\mu_c + \overbrace{\frac12(\gamma-1)^2\sigma_c^2\left[\overset{(1)}{var(\eta_{t+1})}\right] + \frac12(\gamma-1)^2\sigma_c^2\left[\overset{(1)}{var(\eta_{t+2})}\right] + \dots +}^{j}}$$

$$\frac{P_t}{d_t} = \sum_{j=1}^{\infty}\delta^j e^{(\gamma-1)j\mu_c + \frac12(\gamma-1)^2\sigma_c^2 j + j\mu_d + \frac12 j\sigma_d^2 + \frac12 2(\gamma-1)\overset{cov}{\boxed{\sigma_c\sigma_d\rho j}}}$$

$$\delta^j = e^{j\ln\delta}$$

$$\frac{P_t}{d_t} = \sum_{j=1}^{\infty} e^{j\overbrace{\left[\ln\delta - (1-\gamma)\mu_c + \mu_d + \frac12(1-\gamma)^2\sigma_c^2 + \frac12\sigma_d^2 - (1-\gamma)\sigma_c\sigma_d\rho\right]}^{\alpha^*}}$$

$$\frac{P_t}{d_t} = \sum_{j=1}^{\infty} e^{j\alpha^*} \longrightarrow \frac{1}{1-e^{\alpha^*}} - 1 \;;\quad e^{\alpha^*} = e^{\overset{\alpha}{\widetilde{[\cdot]}}}\cdot\overset{\delta}{\cancel{e^{\ln\delta}}}$$

$$\frac{P_t}{d_t} = \frac{1}{1-\delta e^{\alpha}} - 1$$

$$\therefore P_t = d_t\left(\frac{\delta e^{\alpha}}{1-\delta e^{\alpha}}\right) //$$

4. Consider a special case of the model of rational speculative bubbles discussed in this chapter. Assume that infinitely-lived investors are risk-neutral and that there is an asset paying a constant, one-period risk-free return of $R_f = \delta^{-1} > 1$. There is also an infinitely-lived risky asset with price p_t at date t. The risky asset is assumed to pay a dividend of d_t which is declared at date t and paid at the end of the period, date $t+1$. Consider the price $p_t = f_t + b_t$ where

$$f_t = \sum_{i=0}^{\infty}\frac{E_t\left[d_{t+i}\right]}{R_f{}^{i+1}} \tag{1}$$

and

$$b_{t+1} = \begin{cases} \frac{R_f}{q_t}b_t + e_{t+1} & \text{with probability } q_t \\ z_{t+1} & \text{with probability } 1-q_t \end{cases} \tag{2}$$

where $E_t\left[e_{t+1}\right] = E_t\left[z_{t+1}\right] = 0$ and where q_t is a random variable as of date $t-1$ but realized at date t and is uniformly distributed between 0 and 1.

4.a Show whether or not $p_t = f_t + b_t$ subject to the specifications in (1) and (2) is a valid solution for the price of the risky asset.

4.a) $p_t = f_t + b_t$

Check (2) → it should be equal to $E_t[b_{t+1}] = R_f b_t$

$$E_t[b_{t+1}] = \cancel{q_t}\frac{R_f}{\cancel{q_t}}b_t + E_t[\overset{0}{\cancel{e_{t+1}}}]q_t + (1-q_t)\overset{0}{\cancel{E_t[z_{t+1}]}}$$

Since $E_t[e_{t+1}] = E_t[z_{t+1}] = 0$

$\therefore E_t[b_{t+1}] = R_f b_t$ // → valid

4.b Suppose that p_t is the price of a barrel of oil. If $p_t \geq p_{solar}$, then solar energy, which is in perfectly elastic supply, becomes an economically efficient perfect substitute for oil. Can a rational speculative bubble exist for the price of oil? Explain why or why not.

$$E_t[b_{t+1}] = R_f b_t$$

Take lim → $\lim_{i\to\infty} E_t[b_{t+i}] = \begin{cases} +\infty & \text{if } b_t > 0 \\ -\infty & \text{if } b_t < 0 \end{cases}$

However, b_t can't be negative so we only consider the case that bubble is expected to increase infinitely.
If there is a perfect substitute in perfectly elastic supply. Therefore, b_t cannot increase to infinity as p_t cannot rise above p_{solar}.
∴ No bubble for the price of oil.

4.c Suppose p_t is the price of a bond that matures at date $T < \infty$. In this context, the d_t for $t \leq T$ denotes the bond's coupon and principal payments. Can a rational speculative bubble exist for the price of this bond? Explain why or why not.

At maturity date, the bond's price should be equal to $p_T = d_T$ and will become 0 after date T

$$\lim_{i\to\infty} E_t[b_{t+i}] = \begin{cases} +\infty & \text{if } b_t > 0 \\ -\infty & \text{if } b_t < 0 \end{cases}$$

Thus, a rational speculative bubble cannot occur since the bubble path is invalid.

5. Consider an endowment economy with representative agents who maximize the following objective function:

$$\max_{C_s, \{\omega_{is}\}, \forall s, i} E_t\left[\sum_{s=t}^{T} \delta^s u(C_s)\right]$$

where $T < \infty$. Explain why a rational speculative asset price bubble could not exist in such an economy.

At date T, $p_T = f_T = d_T$
$E_t[b_{t+1}] = \delta^{-1} b_t$

If $b_T = 0$, it means that $E_{T-1}[b_T] = E_{T-1}[0] = \delta^{-1} b_{T-1}$ or $b_{T-1} = 0$.
Therefore, $b_t = 0$ for all previous date, $t < T-1$ or a rational speculative asset price bubble couldn't exist.