Quiz 2: Date: May 5, 2022 from 11.00-12.30

Question 1 (40 marks) Score.....................

Consider the Muliperiod model of consumption and portfolio choice. Let an individual in this economy has the utility function as follow:

$$\max_{C_s,\omega_s,\forall_t} E_t\left[\sum_{s=t}^{T-1}\delta^s\left(\frac{C_s^{(1-\gamma)}}{1-\gamma}\right)+\delta^T\left(\frac{W_T^{(1-\gamma)}}{1-\gamma}\right)\right]$$

Assume that there is no wage income $(y_t = 0\,\forall t)$ and a constant risk-free rate return asset , $R_{ft} = R_f$. Also asssume that n=1 and the return of a single risky asset, R_{rt},is independently and identically distributed over time. Denote the proportion of wealth invested in the risky asset at date t as ω_t.

Please read and answer the following questions carefully and completely.

Score.....................

Question 1.1 (10 marks) Derive the first-order condition for the optimal consumption level and portfolio weight at date T-1, C^*_{T-1} and ω^*_{T-1}., and give an explicit expression for C^*_{T-1}

$$S_t = (W_t - C_t) \qquad W_{t+1} = (W_t - C_t)\left(R_{ft} + \sum_{i=1}^{n} w_{it}(R_{it} - R_{ft})\right)$$

$$J(W_{T-1}, T-1) = \max U(C(C_{T-1}, T-1)) + E_{T-1}(B(S_{T-1} R_{T-1}, T))$$

$$\frac{\partial J}{\partial C_{T-1}} = U_c(C(C_{T-1}, T-1)) - E_{T-1}[B_w(W_T, T)R_{T-1}] = 0$$

$$\frac{\partial J}{\partial w_{T-1}} = E_{T-1}[B_W(W_T, T)(R_{i,T-1} - R_{f,T-1})] = 0$$

$$U_C(C_{T-1}, T-1) = R_{f,T-1}[B_W(W_T, T)]$$

$$\delta^{T-1} C_{T-1}^{-\gamma} = \delta^T W_T^{-\gamma}$$

$$C_{T-1}^{-\gamma} = \delta^{T-(T-1)} W_T^{-\gamma}$$

$$\frac{1}{C_{T-1}^{\gamma}} = \delta \frac{1}{W_T^{\gamma}}$$

$$\frac{1}{\delta} = \left(\frac{C_{T-1}}{W_T}\right)^{\gamma}$$

$$\left(\frac{1}{\delta}\right)^{\frac{1}{\gamma}} = \frac{C_{T-1}}{W_T}$$

$$C^*_{T-1} = \left(\frac{1}{\delta}\right)^{\frac{1}{\gamma}} W_T$$

Score.....................

Question 1.2 (10 marks) Solve for the form of J(W_{T-1},T-1).

$$J(W_{T-1}, T-1) = \delta^{T-1} \frac{\left(\left(\frac{1}{\delta}\right)^{\frac{1}{\gamma}} W_T\right)^{1-\gamma}}{1-\gamma} + \delta^T \frac{(S_T + C_T)^{1-\gamma}}{1-\gamma}$$

$$J_W(W_{T-1}, T-1) = U_c(C^*_{T-1}, T-1)$$

$$= \delta^{T-1}\left(\frac{1}{\delta}\right)^{\frac{1}{\gamma}} W_T + \delta^T (S_T + C_T)^{-\gamma}$$

Score.....................

Question 1.3 (10 marks) Derive the first-order condition for the optimal consumption level and portfolio weight at date T-2, C^*_{T-2} and ω^*_{T-2}., and give an explicit expression for C^*_{T-2}

$$\frac{\partial J}{\partial C_{T-2}} = U_C(C_{T-2}, T-2) -$$

$$\frac{\partial J}{\partial W_{T-2}} =$$

$$U_C(C^*_{T-2}, T-2) = E_{T-2}\left[J_W(W_{T-1}, T-1) R_{T-2}\right]$$

$$\delta^{T-2} C_{T-2}^{-\gamma} = \delta^{T-1}\left(\frac{1}{\delta}\right)^{\frac{1}{\gamma}} W_T + \delta^T (S_T + C_T)^{-\gamma} (R_{T-2})$$

$$C^*_{T-2} = \left(\frac{1}{\delta+\delta^2}\right)^{\frac{1}{\gamma}} W_{T-2}$$

Score......................

Question 1.4 (10 marks) Solve for the form of J(W_{T-2},T-2).Based on the pattern for T-1 and T-2, provide expressions for the optimal consumption and portfolio weight at any date T-t, t=1,2,3,....

$$J(W_{T-2}, T-2) = \max_{C_{T-2}, \{w_i, T-2\}} U(C_{T-2}, T-2) + E_{T-2}\left[U(C_{T-1}, T-1) + B(W_T, T)\right]$$

$$\therefore J(W_t, t) = \max_{C_t, \{w_i, t\}} U(C_t, t) + E_t\left[J(W_{t+1}, t+1)\right]$$