

EE325 Section 1 HW 2 Due Thursday February 20th (23:00 hr.), 2020

Use 4 decimal places for numerical answers

1. In Table 1.a. X_i is total microeconomics exam point (total points are 100) and Y_i is GPA of each student.

Table 1.a

Student	Y_i	X_i
1	2.8	63
2	3.4	72
3	3	78
4	3.5	81
5	3.6	87
6	3.0	75
7	2.7	75
8	3.7	90

$$\bar{X} = \frac{\sum X_i}{N} = \frac{621}{8} = 77.625$$

$$\bar{Y} = \frac{\sum Y_i}{N} = \frac{25.7}{8} = 3.2125$$

$$\begin{aligned} \sum (X_i - \bar{X})(Y_i - \bar{Y}) &= (63 - 77.625)(2.8 - 3.2125) + (72 - 77.625)(3.4 - 3.2125) + (78 - 77.625)(3 - 3.2125) \\ &+ (81 - 77.625)(3.5 - 3.2125) + (87 - 77.625)(3.6 - 3.2125) + (75 - 77.625)(3.0 - 3.2125) \\ &+ (75 - 77.625)(2.7 - 3.2125) + (90 - 77.625)(3.7 - 3.2125) = 19.4346 \end{aligned}$$

$$\begin{aligned} \sum (X_i - \bar{X})^2 &= (63 - 77.625)^2 + (72 - 77.625)^2 + (78 - 77.625)^2 + (81 - 77.625)^2 \\ &+ (87 - 77.625)^2 + (75 - 77.625)^2 + (75 - 77.625)^2 + (90 - 77.625)^2 = 511.8748 \end{aligned}$$

1.1 Now consider the two-variable $Y_i = \beta_0 + \beta_1 X_i + u_i$, $u_i \sim NIID(0, \sigma^2)$. Use OLS to find the estimator of β_0 and β_1 . (Note: $NIID$ = Normally, Identically, and Independently Distributed).

$$\hat{\beta}_0 = \bar{Y} - \hat{\beta}_1 \bar{X} = 3.2125 - (0.0341)(77.625) = 0.5655$$

$$\hat{\beta}_1 = \frac{\sum (X_i - \bar{X})(Y_i - \bar{Y})}{\sum (X_i - \bar{X})^2} = \frac{19.4346}{511.8748} = 0.0341$$

$$\hat{Y}_i = 0.5655 + 0.0341 \hat{X}_i + \hat{u}_i$$

1.2 For each observation i , find $\hat{Y}_i$ and $\hat{u}_i$. Show that $\sum_{i=0}^N \hat{u}_i = 0$.

Student	Y_i	X_i
1	2.8	63
2	3.4	72
3	3	78
4	3.5	81
5	3.6	87
6	3.0	75
7	2.7	75
8	3.7	90

$$\sum \hat{u}_i = 0 \quad \sum \hat{Y}_i = 29.7 \quad \sum \hat{X}_i = 437.7$$

from new tables

$$\sum \hat{u}_i = \sum Y_i - \sum \hat{Y}_i$$

$$= 29.7 - 29.7 = 0$$

1.3 Find $var(\hat{u}_i)$, $var(\hat{\beta}_0)$, $var(\hat{\beta}_1)$

$$Var(\hat{u}_i) = \frac{\sum \hat{u}_i^2}{n-2} = \frac{(0.0862)^2 + (1.3799)^2 + (-0.2253)^2 + (1.7224)^2 + (-0.0678)^2 + (1.123)^2 + (1.419)^2 + (-0.0655)^2}{6}$$

$$= \frac{0.0074 + 1.9041 + 0.0508 + 2.9667 + 0.0046 + 1.2611 + 2.0136 + 0.0043}{6}$$

$$= \frac{0.4347}{6} = 0.07245$$

$$Var(\hat{\beta}_1) = \frac{\sigma^2}{SSE} = \frac{\sigma^2}{\sum (x_i - \bar{x})^2} = \frac{0.07245}{511.8748} = 0.00014$$

$$Var(\hat{\beta}_0) = \frac{\sigma^2}{n \sum (x_i - \bar{x})^2} = \frac{43.717(0.0001)}{9(511.8748)} = 0.0119$$

2. Data is listed in the table

X_i	Y_i
10	0
12	2
14	5
16	6
18	7
22	10
24	10
26	15
28	16
30	20

$$\bar{x} = \frac{\sum x_i}{n} = \frac{200}{10} = 20$$

$$\bar{y} = \frac{\sum y_i}{n} = \frac{91}{10} = 9.1$$

$$\sum (x_i - \bar{x})^2 = (10-20)^2 + (12-20)^2 + (14-20)^2 + (16-20)^2 + (18-20)^2 + (22-20)^2 + (24-20)^2 + (26-20)^2 + (28-20)^2 + (30-20)^2 = 440$$

$$\sum (x_i - \bar{x})(y_i - \bar{y}) = (10-20)(0-9.1) + (12-20)(2-9.1) + (14-20)(5-9.1) + (16-20)(6-9.1) + (18-20)(7-9.1) + (22-20)(10-9.1) + (24-20)(10-9.1) + (26-20)(15-9.1) + (28-20)(16-9.1) + (30-20)(20-9.1) = 394$$

2.1 From the simple regression model $Y_i = \beta_0 + \beta_1 X_i + u_i$, $u_i \sim NIID(0, \sigma^2)$. Find estimators of

β_0 and β_1 from the OLS method and interpret the meaning.

$$\hat{\beta}_1 = \frac{\sum (x_i - \bar{x})(y_i - \bar{y})}{\sum (x_i - \bar{x})^2} = \frac{394}{440} \approx 0.8955$$

$$\hat{\beta}_0 = \bar{y} - \hat{\beta}_1 \bar{x} = 9.1 - (0.8955)(20) \approx -8.81$$

When x_i equal to 0, y_i will equal to -8.81 and when x_i increases by one unit y_i increases by 0.8955 units

$$\hat{y}_i = \hat{\beta}_0 + \hat{\beta}_1 \hat{x}_i + \hat{u}_i = -2.81 + 0.2955 x_i$$

2.2 Find the value of $\hat{Y}_i$ and $\hat{u}_i$. Show that $\sum_{i=0}^N \hat{u}_i = 0$.

$$\hat{u}_i = y_i - \hat{y}_i = y_i - (-2.81 + 0.2955 x_i)$$

from new table

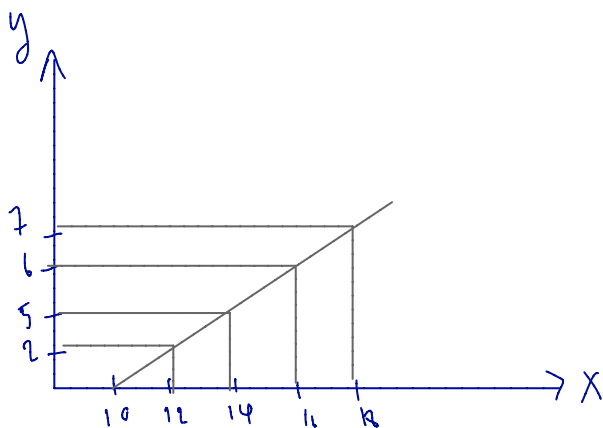
$$\begin{aligned} E\hat{u}_i &= E y_i - E \hat{y}_i \\ &= 91 - 91 = 0 \end{aligned}$$

X_i	Y_i	$\hat{u}_i$	$\hat{y}_i$	X_i^2
10	0	-0.145	0	100
12	2	0.064	2	144
14	5	1.273	5	196
16	6	0.482	6	256
18	7	-0.509	7	324
22	10	-0.991	10	484
24	10	-2.132	10	576
26	15	0.522	15	676
28	16	-0.264	16	784
30	20	1.945	20	900

$$\begin{array}{r} \text{total} \\ \hline 0 \\ \hline 91 \\ \hline 4440 \end{array}$$

2.3 Plot graph and draw regression line. Does the line pass $(\bar{X}, \bar{Y})$?

$$\bar{y} = \hat{\beta}_0 + \hat{\beta}_1 \bar{x} = -2.81 + 0.2955(20) = 5.1 \rightarrow \text{pas } (\bar{x}, \bar{y})$$



2.4 If $X_i = 16$, what is the predicted Y ?

$$\hat{y}_i = -2.81 + 0.2955(16) = 5.518$$

2.5 Find $\text{var}(\hat{u}_i)$, $\text{var}(\hat{\beta}_0)$, $\text{var}(\hat{\beta}_1)$

$$\begin{aligned} \text{Var}(\hat{u}_i) &= \frac{E u_i^2}{n-2} = \frac{0.021 + 0.0041 + 1.6205 + 0.2323 + 0.0055 + 0.7939}{8} \\ &= \frac{2.1451 + 0.2777 + 0.0197 + 3.713}{8} = \frac{14.099}{8} = 1.7619 = \hat{\sigma}^2 \end{aligned}$$

$$\text{Var}(\hat{\beta}_0) = \frac{E x_i^2}{n E c x_i - \bar{x}^2} \hat{\sigma}^2 = \frac{4440}{10(440)} (1.7619) = 1.7771$$

$$\text{Var}(\hat{\beta}_1) = \frac{\hat{\sigma}^2}{SST} = \frac{\hat{\sigma}^2}{E c x_i - \bar{x}^2} = \frac{1.7619}{440} = 0.004$$

3. Consider the below regression function:

$$Y_i = \beta_1 X_i + u_i,$$

Where $u_i \sim NIID(0, \sigma^2)$. Find an OLS estimator of β_1 . Then, provide a proof that this is an unbiased estimator. Please state the SLR assumption(s) when used.

OLS estimator of β_1

$$s = \sum_i (u_i)^2$$

↪ minimize

$$\frac{d}{d\beta_1} \sum_i (Y_i - \beta_1 X_i)^2 = 0$$

$$\sum_i 2(Y_i - \beta_1 X_i)(-X_i) = 0$$

$$\sum_i (Y_i - \beta_1 X_i) X_i = 0$$

$$\sum_i (Y_i X_i - \beta_1 X_i^2) = 0$$

$$\sum_i X_i Y_i - \sum_i \beta_1 X_i^2 = 0$$

$$\beta_1 = \frac{\sum_i X_i Y_i}{\sum_i X_i^2}$$

proof: substitute $y_i = \beta_1 x_i + u_i$ in $\hat{\beta}_1$ equation

$$\hat{\beta}_1 = \frac{\sum_i X_i (\beta_1 X_i + u_i)}{\sum_i X_i^2}$$

$$= \frac{\sum_i (\beta_1 X_i^2 + u_i X_i)}{\sum_i X_i^2}$$

$$\hat{\beta}_1 = \beta_1 + \frac{\sum_i u_i X_i}{\sum_i X_i^2}$$

$$E(\hat{\beta}_1) = E(\beta_1) + E\left(\frac{\sum_i u_i X_i}{\sum_i X_i^2}\right)$$

SLR 4 $\nearrow E(u_i | X_i) = 0$

$$E(\hat{\beta}_1) = \beta_1$$