Last time

$$\dot{y}(t) + u(t)\,y(t) = w(t) \qquad -(12)$$

Sol^n

$$y(t) = e^{-\int u(t)\,dt}\left(A + \int w(t)\,e^{\int u(t)\,dt}\,dt\right) \qquad -(15)$$

2.14 <u>Exact Differential Equation</u>

$$g(y,t)\frac{dy}{dt} + f(y,t) = 0 \qquad -(16)$$

$$\frac{dy}{dt} = F(y,t) \rightarrow \frac{\partial F}{\partial t} = f(y,t)$$

$$\frac{\partial F}{\partial y} = g(y,t)$$

$$dF = g(y,t)\,dy + f(y,t)\,dt$$

$$\frac{dF}{dt} = g(y,t)\frac{dy}{dt} + f(y,t) = \underbrace{0}_{\text{from (16)}}$$

$$F(y,t) = C$$

Let $\frac{\partial F}{\partial y} = M$

$$F(y,t) = \int M\,dy + \psi(t)$$

$$\boxed{C = \int M\,dy + \psi(t)}$$

Ex $2ty\,dy + y^2\,dt = 0$

$$g(y,t) = 2ty = \frac{\partial F}{\partial y} \longrightarrow \frac{\partial^2 F}{\partial y\,\partial t} = 2y = \frac{\partial g}{\partial t} \qquad -(a)$$

$$f(y,t) = y^2 = \frac{\partial F}{\partial t} \longrightarrow \frac{\partial^2 F}{\partial t\,\partial y} = 2y = \frac{\partial f}{\partial y} \qquad -(b)$$

Let $g(y,t) = \boxed{\frac{\partial F}{\partial y} = M}$

step (i) $F(y,t) = \int M\,dy + \psi(t)$

$= \int 2ty\,dy + \psi(t)$

$F(y,t) = y^2 t + \psi(t) + C_1$ — $(\star)$

step (ii) from $(\star)$ $\frac{\partial F(y,t)}{\partial t} = y^2 + \psi'(t)$ — $(\star\star)$

since from (b) $f(y,t) = y^2 = \frac{\partial F}{\partial t}$ — $(\star\star\star)$

$(\star\star\star) = (\star\star)$

$\therefore$ $y^2 = y^2 + \psi'(t)$ —

we need $\psi'(t) = 0$

step iii $\psi(t) = \int \psi'(t)\,dt$

$= \int 0\,dt$

$\psi(t) = C_2$

step iv

go back to step (i) and sub $\psi(t) = C_2$

$F(y,t) = y^2 t + C_2 + C_1$

$F(y,t) = y^2 t + C_3$; $C_2 + C_1 = C_3$

And we know that $F(y,t) = C$

$\therefore$ $y^2 t + C_3 = C$

$y^2 t = C_4$; $C_4 = C - C_3$

$y^2 = C_4 t^{-1}$

$\boxed{y(t) = C_5 t^{-\frac{1}{2}}}$ where $C_5 = (C_4)^{\frac{1}{2}}$

this example $2ty\frac{dy}{dt} + y^2 = 0$ CHECK YOUR ANSWER!

Also we can get (15) by using the concept of Exact DE

$$\frac{dy(t)}{dt} + u(t)\,y(t) = w(t)$$

$$\frac{dy}{dt} + (uy - w) = 0 \qquad \text{———} \quad (*)$$

From $g(y,t)\frac{dy}{dt} + f(y,t) = 0$

$$g(y,t) = 1 = \frac{\partial F}{\partial y} \longrightarrow \frac{\partial^2 F}{\partial y\,\partial t} = 0$$

$$f(y,t) = u(t)\,y(t) - w(t) = \frac{\partial F}{\partial t} \longrightarrow \frac{\partial^2 F}{\partial t\,\partial y} = u(t)$$

Not exact

we can make $(*)$ exact DE by premultiplying by '$I(t)$'

$I(t)$ = an integrating factor

$(*)$ becomes $\underbrace{I(t)}_{=g(\cdot)}\frac{dy}{dt} + \underbrace{I(t)\left(u(t)\,y(t) - w(t)\right)}_{=f(\cdot)} = 0$

$$\frac{\partial g}{\partial t} = \frac{\partial I(t)}{\partial t} \qquad \frac{\partial g}{\partial t} = \frac{\partial f}{\partial y}$$

$$\frac{\partial f}{\partial y} = I(t)\,u(t) \qquad \therefore \frac{\partial I(t)}{\partial t} = I(t)\,u(t) \Rightarrow \frac{1}{I}\frac{dI}{dt} = u(t)$$

$$\int \frac{1}{I}\frac{dI}{\partial t}dt = \int u(t)\,dt$$

$$\ln|I| + c = \int u(t)\,dt$$

$$\ln|I| = -c + \int u(t)\,dt$$

$$e^{\ln|I|} = e^{-c + \int u(t)\,dt}$$

$$\boxed{I(t) = A\,e^{\int u(t)\,dt}} \quad ; \quad A = e^{-c}$$

Assuming that $A = 1$, $I(t) = e^{\int u(t)\,dt}$

$$\therefore \quad \underbrace{e^{\int u(t)\,dt}}\frac{dy}{dt} + \underbrace{e^{\int u(t)\,dt}\left[u(t)\,y(t) - w(t)\right]} = 0$$

$$M = g(y,t) = \frac{\partial F}{\partial y} \qquad f(y,t) = \frac{\partial F}{\partial t}$$

Step (i)

$$F(y,t) = \int M\,dy + \psi(t) \qquad \text{— (A1)}$$

$$= \int e^{\int u(t)\,dt}\,dy + \psi(t)$$

$$= e^{\int u(t)\,dt}\,y + c + \psi(t) \qquad \text{— (A2)}$$

step (ii)

diff (A2) w.r.t 't'

$$\frac{\partial F}{\partial t} = y\,u\,e^{\int u(t)\,dt} + \psi'(t) \qquad \text{— (A3)}$$

since $f(y,t) = \frac{\partial F}{\partial t} = e^{\int u(t)\,dt}\left[u.y - w\right]$

$$\therefore \quad \underbrace{y\,u\,e^{\int u(t)\,dt}} + \psi'(t) = \underbrace{e^{\int u(t)\,dt}\left[u\,y - w\right]}$$

$$\psi'(t) = -w\,e^{\int u(t)\,dt} \qquad \text{— (A4)}$$

step (iii)

Find $\psi(t) = \int \psi'(t)\,dt$

From (A4)

$$\int \psi'(t)\,dt = -\int w(t)\,e^{\int u(t)\,dt}\,dt$$

$$\psi(t) + C_2 = -\int w(t)\,e^{\int u(t)}\,dt \qquad \text{— (A5)}$$

Step iv

(A5) $\rightarrow$ (A2)

$$F(y,t) = y\,e^{\int u(t)\,dt} + C_1 - \int w(t)\,e^{\int u(t)\,dt}\,dt - C_2$$

$$C = ye^{\int u(t)\,dt} - \int w(t)\, e^{\int u(t)\,dt}\,dt + (C_1 - C_2)$$

$$ye^{\int u(t)\,dt} = \left[C - (C_1 - C_2)\right] + \int w(t)\, e^{\int u(t)\,dt}\,dt$$

$$\boxed{y(t) = e^{-\int u(t)\,dt}\left[A + \int w(t)\, e^{\int u(t)\,dt}\,dt\right]} \quad — (15)$$

$$A = C - C_1 + C_2$$

Ex Find the general solⁿ of $\frac{dy}{dt} + 2ty = t$

$u(t) = 2t$ $\qquad\qquad$ $w(t) = t$

$u(t) = 2t \Rightarrow \int u(t)\,dt = t^2 + C_1$

$$y(t) = e^{-(t^2+C_1)}\left[A + \int t\, e^{(t^2+C_1)}\,dt\right]$$

$$= e^{-t^2} e^{-C_1}\left[A + e^{C_1}\int t\, e^{t^2}\,dt\right]$$

$$= e^{-t^2} e^{-C_1} A + e^{-t^2}\int t e^{t^2}\,dt$$

$$= e^{-t^2} e^{-C_1} A + e^{-t^2}\left(\frac{1}{2} e^{t^2} + C_2\right)$$

$$= A e^{-C_1} e^{-t^2} + C_2 e^{-t^2} + \frac{1}{2}$$

$$y(t) = B e^{-t^2} + \frac{1}{2} \qquad\qquad B = A e^{-C_1} + C_2$$

$y'(t) = ?$ $\quad$ CHECK YOUR ANSWER

Ex Wealth Accomolation

$\tilde{w}(t)$ = the amount of savings in account @ time t

$\tilde{y}(t)$ = the deposit rate

$c(t)$ = the withdrawal rate

$r(t)$ = the continuous compounding interest rate

$$\frac{d\tilde{w}(t)}{dt} = \underbrace{r(t)}_{u(t) = -r(t)}\tilde{w}(t) + \underbrace{\tilde{y}(t) - c(t)}_{w(t)}$$

$$\frac{dy}{dt} + u(t)\,y(t) = w(t)$$

$$\tilde{w}(t) = e^{-\int -r(t)\,dt}\left[A + \int (y(t) - c(t))\,e^{\int -r(t)\,dt}\,dt\right]$$

2.1.5 Separable Differential Equations

Not Exact DE

$F(y,t) = C$

$$\frac{dy}{dt} = F(y,t) = f(t)\,.\,g(y) \quad \text{---(1)}$$

We say that this DE is separable

i.e $\dfrac{dy}{dt} = -2ty^2 \rightarrow f(t) = -2t$

$g(y) = y^2$

$\dfrac{dy}{dt} = (t^2 + t)y^2 \rightarrow f(t) = t^2 + t$

$g(y) = y^2$

$\dfrac{dy}{dt} = y^2 + t^3 \Rightarrow$ Non separable

$\dfrac{dy}{dt} = yt + t^2 \Rightarrow$ Non-separable.

A general method for solving (1) can be expressed as "4-Steps method"

step (i) write (1) as $\dfrac{dy}{dt} = f(t)\,.\,g(y)$

step (ii) separate the functions $\dfrac{1}{g(y)}\dfrac{dy}{} = f(t)\,.\,dt$

step (iii) Integrating each side $\int \frac{1}{g(y)} dy = \int f(t)\, dt$

step (iv) evaluating two integrals

and then solve for $y(t)$ explicitly if ~~possible~~ possible

→ possibly in an implicit form

If $g(y)$ has a "zero" value at $y = a$

$[g(y(t)) = 0$ @ $y(t) = a]$

$y(t) = a$ is also a sol^{n}