CHAPTER 4

Basic Matrix Algebra and Applications

Topics: **Basic Matrix Algebra**

Outline:

- ☺ Representation of system of equation by matrix notation
- ☺ Multiplication of matrices
- ☺ Determinant and singularity of matrix
- ☺ Matrix Inversion
- ☺ Cramer's rule

Matrix Algebra

Matrix is a rectangular array of numbers, parameters, or variables

$$\begin{matrix} ☺ & ☺ & ☺ & \cdots & ☺ \\ ☺ & ☺ & ☺ & \cdots & ☺ \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ ☺ & ☺ & ☺ & \cdots & ☺ \end{matrix}$$

$$\begin{bmatrix} \spadesuit & \heartsuit \\ \clubsuit & \diamondsuit \end{bmatrix}, \begin{pmatrix} \spadesuit & \heartsuit \\ \clubsuit & \diamondsuit \end{pmatrix}, \begin{Vmatrix} \spadesuit & \heartsuit \\ \clubsuit & \diamondsuit \end{Vmatrix}$$

$$\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}_{m\times n}$$

$$[a_{ij}]_{m\times n}, i = 1, \dots, m; j = 1, \dots, n$$

<u>Benefits of using Matrix</u>

1. Matrix provides a compact way of writing an equation system, even an extremely large one

2. Matrix provides a way of testing the existence of a solution by evaluation of a determinant.

3. Matrix can be used to find that solution if exists. This is done by using Inverse matrix or Cramer's rule

Limitation: matrix algebra is applicable only to linear-equation systems.
In some cases, we can transform variables so as to obtain a linear relation to work with. For example, the nonlinear function $y = ax^b$ can be transformed into:

………………………………………………………………………………………

A compact way of writing an equation system:

A system of m linear equations with n endogenous variables:

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= d_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= d_2 \\ \vdots \qquad & \quad \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= d_m \end{aligned}$$

In matrix form:

$$Ax = d$$

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}_{m\times n} \qquad x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}_{n\times 1} \qquad d = \begin{bmatrix} d_1 \\ d_2 \\ \vdots \\ d_m \end{bmatrix}_{m\times 1}$$

Note: Terminology

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}_{n\times 1}$$: column vector

$$z = \begin{pmatrix} z_1 & z_2 & \cdots & z_n \end{pmatrix}_{1\times n}$$: row vector

Recapping some Matrix definitions

Definition 1

Two matrices $\mathbf{A} = [a_{ij}] \in \mathbf{R}^{m\times n}$ and $\mathbf{B} = [b_{ij}] \in \mathbf{R}^{p\times q}$ are equal, written $\mathbf{A} = \mathbf{B}$ if m=p and n=q, and $a_{ij} = b_{ij}$ for all i and j

Example $A = \begin{bmatrix} 4 & 2 \\ 1 & 3 \end{bmatrix}$ $B = \begin{bmatrix} 4 & 2 \\ 3 & 1 \end{bmatrix}$ $C = \begin{bmatrix} 4 & 2 \\ 1 & 3 \end{bmatrix}$

Definition 2

Given a matrix $\mathbf{A}=[a_{ij}]\in\mathbf{R}^{m\times n}$ and a scalar $c\in\mathbf{R}$ the scalar multiplication of matrix $\mathbf{A}$ by the scalar c is given by $c\mathbf{A}=[ca_{ij}]\in\mathbf{R}^{m\times n}$

Example

$$7\begin{bmatrix}3 & -1\\ 0 & 5\end{bmatrix}=$$

Definition 3

The additional of two matrices $\mathbf{A}=[a_{ij}]\in\mathbf{R}^{m\times n}$ and $\mathbf{B}=[b_{ij}]\in\mathbf{R}^{p\times q}$ is defined only if m=p And n=q, and is given by

$$\mathbf{A}+\mathbf{B}=[a_{ij}]+[b_{ij}]=[a_{ij}+b_{ij}]\in\mathbf{R}^{m\times n}$$

Example

$$\begin{bmatrix}4 & 9\\ 2 & 1\end{bmatrix}+\begin{bmatrix}2 & 0\\ 0 & 7\end{bmatrix}=$$

Transposition of a Matrix

Definition 6

Let $\mathbf{A}=[a_{ij}]\in\mathbf{R}^{m\times n}$,The transpose of $\mathbf{A}$ is given by,

$$\mathbf{A}^T=[a_{ij}^{\ T}]\in\mathbf{R}^{n\times m}, where\, a_{ij}^T=a_{ji}$$

When a matrix is transposed, the rows becomes columns and vice versa

$$A=\begin{pmatrix}a_{11} & a_{12} & a_{13}\\ a_{21} & a_{22} & a_{23}\end{pmatrix}_{2\times 3}$$

$$A^T=\begin{pmatrix}a_{11} & a_{21}\\ a_{12} & a_{22}\\ a_{13} & a_{23}\end{pmatrix}_{3\times 2}$$

Example

$$A=\begin{bmatrix}3 & 8 & 9\\ 1 & 0 & 4\end{bmatrix} \qquad A'=$$

$$B=\begin{bmatrix}3 & 4\\ 1 & 7\end{bmatrix} \qquad B'=$$

$$(AB)^T=B^TA^T$$

Recapping some special types of matrices

\>> Square matrix

A matrix $\mathbf{A} = [a_{ij}] \in \mathbf{R}^{m \times n}$ is a square matrix if m=n. That is, the number of rows is equal to the number of columns

Example

$$A = \begin{bmatrix} 1 & 3 & 4 \\ 2 & 4 & 5 \\ 5 & 2 & 9 \end{bmatrix} \quad B = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix}$$

\>> Diagonal matrix

The diagonal elements of a square matrix $\mathbf{A} \in \mathbf{R}^{m \times n}$ are the elements

$$a_{11}, a_{22}, \ldots, a_{nn}$$

A diagonal matrix is a square matrix

$$\mathbf{A} = [a_{ij}] \in \mathbf{R}^{n \times n}$$

where $a_{ij} = 0 \text{ if } i \neq j$

$$\text{เช่น } I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad B = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

\>> Identity matrix

An identity matrix is a diagonal matrix $\mathbf{I}_n$ of order nxn whose diagonal elements being ones. An identity matrix is thus necessarily square.

$$I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}_{2\times 2} \qquad I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}_{3\times 3}$$

\>> Null matrix

A matrix $\mathbf{A} = [a_{ij}] \in \mathbf{R}^{m \times n}$ is a zero matrix/null matrix if all of its elements are zeros. We write

$$\mathbf{A} = \mathbf{0}$$

Example

$$A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

>>Triangular matrix

A matrix "A" is **upper triangular** if it is square and $a_{ij} = 0 \; if \; i > j$, and **lower triangular** if it is square and $a_{ij} = 0 \; if \; i < j$

$$L = \begin{bmatrix} 2 & 0 & 0 \\ 1 & 3 & 0 \\ 6 & 5 & 4 \end{bmatrix} \qquad U = \begin{bmatrix} 2 & 1 & 6 \\ 0 & 3 & 5 \\ 0 & 0 & 4 \end{bmatrix}$$

>> Symmetric matrix

A matrix is symmetric if $A^T = A$. That is $a_{ij} = a_{ji} \quad for \; all \; i, j$ เช่น

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{bmatrix}$$

<u>We will learn the following matrix operation (Matrix Algebra)</u>

- Multiplication of matrices
- Determinant of a matrix
- Inverses

Multiplication of matrices

The Multiplication of matrix $A = \left[a_{ij} \right]_{m \times n}$ by matrix $B = \left[b_{ij} \right]_{p \times q}$ is defined only if $n = p$ and is given by $AB = \left[\sum_{k=1}^{n} a_{ik} b_{kj} \right]_{m \times q}$

....

....

....

....

....

....

Example

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \end{bmatrix} = \begin{bmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{bmatrix}$$

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \end{bmatrix} = \begin{bmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{bmatrix}$$

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \end{bmatrix} = \begin{bmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{bmatrix}$$

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} 5 & 6 \\ 7 & 8 \end{pmatrix} =$$

Question: $A_{4\times 3} B_{2\times 4} = ?$

"dot product"

Definition 5

When a row vector is multiplied by a column vector, the product is called dot product or inner product. That is, if

$$\mathbf{a}\,\mathbf{b} = \begin{bmatrix} a_1 & a_2 & \dots & a_n \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix} = \left[\sum_{i=1}^{n} a_i b_i \right] \in \mathbf{R}^{1\times 1} = \mathbf{R},$$

$$u = \begin{bmatrix} 3 & 4 \end{bmatrix}_{1\times 2} \quad v = \begin{bmatrix} 9 \\ 7 \end{bmatrix}_{2\times 1} \qquad uv = \begin{bmatrix} 3*9 \; + \; 4*7 \end{bmatrix} = \begin{bmatrix} 55 \end{bmatrix} = 55$$

Note: A matrix is idempotent if $A^2 = A$

……………………………………………………………………………………

$$A = \begin{pmatrix} 5 & -5 \\ 4 & -4 \end{pmatrix}$$

Determinant

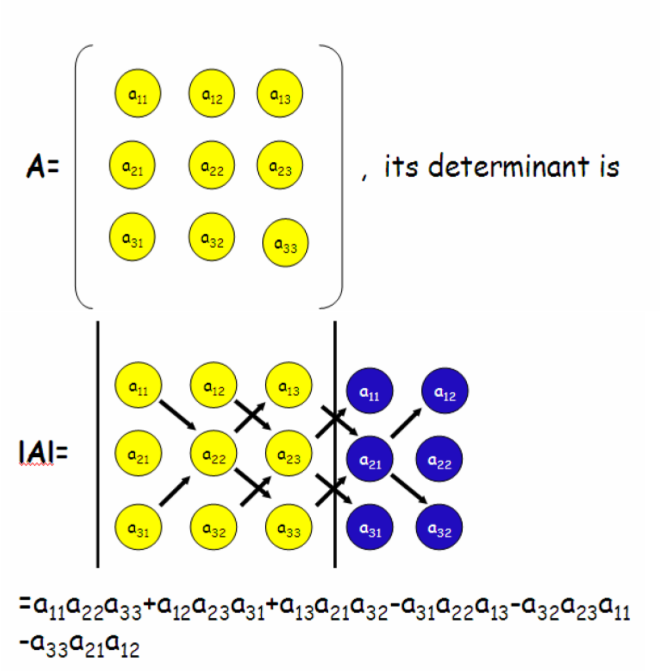
The determinant of a square matrix A, denoted by $|A|$, is a **uniquely defined number** associated with that matrix

For 2 by 2 matrix $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$, <u>the 2nd order determinant</u> is

$$|A| = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$|A| = \begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} =$$

Evaluating a Third-Order Determinant



$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} =$$

<u>n[th]-order determinant by Laplace Expansion</u>

Let A be a matrix in $R^{n\times n}$. The determinant of A or det A is given by $|A|$.

$$|A| = \begin{cases} \sum_{j=1}^{n} a_{ij}C_{ij} & i = 1,2,\ldots,n \text{ (expansion by i}^{\text{th}}\text{row)} \\ \sum_{i=1}^{n} a_{ij}C_{ij} & j = 1,2,\ldots,n \text{ (expansion by j}^{\text{th}}\text{column)} \end{cases}$$

If we choose row i,

$$\begin{bmatrix} a_{11} & a_{12} & a_{j1} & a_{1n} \\ a_{21} & a_{22} & a_{j2} & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{i1} & a_{i2} & a_{ij} & a_{in} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & a_{nj} & a_{nn} \end{bmatrix}_{n\times n}$$

If we choose column j,

$$\begin{bmatrix} a_{11} & a_{12} & a_{j1} & a_{1n} \\ a_{21} & a_{22} & a_{j2} & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{i1} & a_{i2} & a_{ij} & a_{in} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & a_{nj} & a_{nn} \end{bmatrix}_{n\times n}$$

>> We call C_{ij}, Cofactor of $element\ a_{ij}$.

>> C_{ij} can be found by $(-1)^{i+j}M_{ij}$

The minor of element a_{ij}, M_{ij}, is the determinant of the matrix resulting from deleting row i and column j **of matrix $A_{n\times n}$. In other words,** a "minor" is the determinant of the square matrix formed by deleting one row and one column from some larger square matrix.

Example, find the following determinant:

$$\begin{vmatrix} 15 & 7 & 9 \\ 2 & 5 & 6 \\ 9 & 0 & 12 \end{vmatrix}$$

Properties of Determinants

Property I

The interchange of rows and columns does not affect the value of a determinant. In other words, the determinant of a matrix A has the same value as that of its transpose A', that is

Example

$A = \begin{bmatrix} 4 & 3 \\ 5 & 6 \end{bmatrix}$ detA=

$B = \begin{bmatrix} 4 & 5 \\ 3 & 6 \end{bmatrix}$ detB=

Property II

The interchange of any two rows (or any two columns) will alter the sign, but not the numerical value, of the determinant

Example

$A=\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ detA=

$B=\begin{bmatrix} c & d \\ a & b \end{bmatrix}$ detB=

Property III

The multiplication of any one row (or one column) by a scalar k will change the value of the determinant k-fold

Example

$A=\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ detA=

$B=\begin{bmatrix} ka & kb \\ c & d \end{bmatrix}$ detB=

Property IV

The addition (subtraction) of a multiple of any row to (form) another row will leave the value of the determinant unaltered

Example

$$A=\begin{bmatrix} a & b \\ c & d \end{bmatrix} \qquad detA=$$

$$B=\begin{bmatrix} a & b \\ c+ka & d+kb \end{bmatrix} \qquad detB=$$

Property V

If one row (or column) is a multiple of another row (or column), the value of the determinant will be zero

Example

A= $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ detA=

B= $\begin{bmatrix} 2a & 2b \\ a & b \end{bmatrix}$ detB=

The determinant & Non-singularity

If $|A| = 0$, square matrix $A_{n\times n}$ is a singular Matrix

: A square matrix that is not invertible is called **singular** or **degenerate**. A square matrix is singular if and only if its determinant is 0

……………………………………………………………………………………

If $|A| \neq 0$, matrix A is. a non-singular Matrix

……………………………………………………………………………………

The inverse of A

Assume that an $n \times n$ *nonsingular matrix A* is given by:

$$\begin{bmatrix} a_{11} & a_{12} & a_{j1} & a_{1n} \\ a_{21} & a_{22} & a_{j2} & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{i1} & a_{i2} & a_{ij} & a_{in} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & a_{nj} & a_{nn} \end{bmatrix}_{n\times n}$$

$$A^{-1} = \frac{1}{|A|} \cdot adjA$$

To find the inverse of matrix A:

1.) Find $\det A, |A|$, if $|A| = 0$, A is a singular matrix and the inverse of matrix A cannot be found.

2.) Find Minor (M_{ij}) for all element of the matrix in row $i = 1, \dots, n$ and column $j = 1, \dots, n$

3.) Find Cofactor (C_{ij}) from Minor (M_{ij}) for all elements of the matrix in row $i = 1, \dots, n$ and column $j = 1, \dots, n$

4.) Get the Cofactor matrix C in which each element is Cofactor (C_{ij}) of element a_{ij}

5.) The transpose of C is called Adjoint A (adjA)

$$a\text{djA} = \left[C_{ij}\right]^{\text{T}}$$

6.) Multiple$adjA$ by scalar $\frac{1}{\det A}$ adjA and we will get the inverse of matrix A, A^{-1} .

Properties of the inverse matrix A^{-1}

I.……………………………………

II.……………………………………..

III.……………………………………..

IV.……………………………………

Find A^{-1}

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$A = \begin{bmatrix} 3 & 2 \\ 1 & 0 \end{bmatrix}$$

$$A = \begin{pmatrix} 15 & 7 & 9 \\ 2 & 5 & 6 \\ 9 & 0 & 12 \end{pmatrix}$$

To find the solution for system of equation by using matrix

We can use (1.) Inverse matrix (2.) Cramer's rule

Remind that:

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= d_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= d_2 \\ \vdots \qquad\qquad\qquad & \quad \vdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n &= d_n \end{aligned}$$

We can write this system of equations in the form of matrix as:

…………………………………………………………………………………………

…………………………………………………………………………………………

…………………………………………………………………………………………

…………………………………………………………………………………………

…………………………………………………………………………………………

…………………………………………………………………………………………

>> Use inverse matrix

…………………………………………………………………………………………
…………………………………………………………………………………………
…………………………………………………………………………………………
…………………………………………………………………………………………
…………………………………………………………………………………………
…………………………………………………………………………………………
…………………………………………………………………………………………
…………………………………………………………………………………………
…………………………………………………………………………………………

>> Use Cramer's Rule:

From,

$$\boxed{X_{n\times 1} = A_{n\times n}^{-1} d_{n\times 1}}$$

That is,

$$X = \frac{1}{|A|}\begin{pmatrix} C_{11} & C_{12} & \cdots & C_{1n} \\ C_{21} & C_{22} & \cdots & C_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ C_{n1} & C_{n2} & \cdots & C_{nn} \end{pmatrix}^T \begin{bmatrix} d_1 \\ \vdots \\ d_n \end{bmatrix}$$

$$; C_{ij} = (-1)^{i+j} M_{ij} \quad \textit{and}$$

$$|A| = \begin{cases} \sum_{j=1}^{n} a_{ij}C_{ij} & i = 1,2,\ldots,n \text{ (expansion by i}^{\text{th}}\text{row)} \\ \sum_{i=1}^{n} a_{ij}C_{ij} & j = 1,2,\ldots,n \text{ (expansion by j}^{\text{th}}\text{column)} \end{cases}$$

$$X = \frac{1}{|A|}\begin{pmatrix} C_{11} & C_{21} & \cdots & C_{n1} \\ C_{12} & C_{22} & \cdots & C_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ C_{1n} & C_{2n} & \cdots & C_{nn} \end{pmatrix} \begin{bmatrix} d_1 \\ \vdots \\ d_n \end{bmatrix}$$

$$X=\begin{pmatrix}\frac{1}{|A|}\left(C_{11}d_1+C_{21}d_2+\cdots+C_{n1}d_n\right)\\ \frac{1}{|A|}\left(C_{12}d_1+C_{22}d_2+\cdots+C_{n2}d_n\right)\\ \vdots\\ \frac{1}{|A|}\left(C_{1n}d_1+C_{2n}d_2+\cdots+C_{nn}d_n\right)\end{pmatrix}=\begin{pmatrix}\frac{1}{|A|}\sum_{i=1}^{n}C_{i1}d_i\\ \frac{1}{|A|}\sum_{i=1}^{n}C_{i2}d_i\\ \vdots\\ \frac{1}{|A|}\sum_{i=1}^{n}C_{in}d_i\end{pmatrix}$$

$C_{11}d_1+C_{21}d_2+\cdots+C_{n1}d_n$ is the determinant of matrix A when replacing column 1 of matrix *A* by vector *d. That is,*

$$C_{11}d_1+C_{21}d_2+\cdots+C_{n1}d_n=\begin{vmatrix}d_1 & a_{12} & \cdots & a_{1n}\\ d_2 & a_{22} & \cdots & a_{2n}\\ \vdots & \vdots & \ddots & \vdots\\ d_n & a_{n2} & \cdots & a_{nn}\end{vmatrix}=|A_1|$$

Likewise,

$$\sum_{i=1}^{n}C_{i2}d_i=\begin{vmatrix}a_{11} & d_1 & \cdots & a_{1n}\\ a_{21} & d_2 & \cdots & a_{2n}\\ \vdots & \vdots & \ddots & \vdots\\ a_{n1} & d_n & \cdots & a_{nn}\end{vmatrix}=|A_2|$$

$$\sum_{i=1}^{n}C_{in}d_i=\begin{vmatrix}a_{11} & a_{12} & \cdots & d_1\\ a_{21} & a_{22} & \cdots & d_2\\ \vdots & \vdots & \ddots & \vdots\\ a_{n1} & a_{n2} & \cdots & d_n\end{vmatrix}=|A_n|$$

Therefore, Cramer's rule is

………………………………………………………………………………………

………………………………………………………………………………………

………………………………………………………………………………………

………………………………………………………………………………………

………………………………………………………………………………………

………………………………………………………………………………………

………………………………………………………………………………………

………………………………………………………………………………………

Example,

$Eq.1 \quad 5x + 3y = 30$

$Eq.2 \quad 6x - 2y = 8$

Find the solution to this system of equations by (1.) Inverse matrix (2.) Cramer’s rule