EE 415 / 418 Game Theory (2/2011)

Lecture 9. Repeated Games continued

- Infinitely Repeated Games
- Friedman’s Theorem or Folk Theorem
- Read topic 2.3B

Infinitely Repeated Games

- In finite-stage repeated game, it is shown that credible treats or promises about future behavior can influence current behavior. In other words, condition your play on the past play of your opponent can lead to equilibrium outcomes that do not arise when the game is played only once.
- If there are multiple equilibria of the stage game G, then there may be subgame-perfect outcome of repeated game $G(T)$ in which, for any $t < T$, the outcome of stage t is not a Nash equilibrium of G. (we can support M cooperation and reward as SPNE)
- This section shows that the stronger result holds in infinitely repeated games.
- There may be subgame-perfect outcome of infinitely repeated game in which no stage's outcome is a Nash equilibrium, even if the stage game has a unique Nash equilibrium.

Infinitely Repeated Games

- Definition
  - Given a stage game G, let $G(\infty, \delta)$ denote the infinitely repeated game in which G is repeated forever and players share the discount factor δ.
  - For each t, the outcome of the $t - 1$ preceding plays of the stage game are observed before the tth stage begins.
  - Each player's payoff in $G(\infty, \delta)$ is the present value of the player's payoffs from the infinite sequence of stage games.

Infinitely Repeated Games

- Suppose the Prisoners' Dilemma in Figure 6 be repeated infinitely and that, for each t, the outcome of the t - 1 preceding plays of the stage game are observed before the t^{th} stage begins.

	L_2	R_2
L_1	1, 1	5, 0
R_1	0, 5	4, 4

Figure 6

Infinitely Repeated Games

- In this game, a simple sum of each stage payoff does not provide a useful measure of a player's payoff from the entire game since no matter how the game be played, the sum of payoffs is infinity.
- Let $\delta = 1/(1 + r)$ denote the discount factor being the value today of a dollar to be received one stage later, where r is the interest rate per stage.

Infinitely Repeated Games

- The measure of a player's payoff from the entire game is given by the present value of the payoffs

$$\pi_1 + \delta^1 \pi_2 + \delta^2 \pi_3 + \cdots = \sum_{t=1}^{\infty} \delta^{t-1} \pi_t$$

where π_t denote the payoff from playing stage t of the repeated game.

Infinitely Repeated Games

- One can use δ to reinterpret an infinitely repeated game as a repeated game that ends after a random number of rounds.
- Let p be the probability that the game ends in the current round so that there is a probability of $1-p$ that the game will be played for at least one more round.
- Then in round t the payoff to be receive in the $t+1$ th round is worth $(1-p)/(1+r)\ \pi_{t+1}$.
- Letting $\delta = (1-p)/(1+r)$. Then the present value $\sum_{t=1}^{\infty} \delta^{t-1}\pi_t$

  reflects both time-value of money and the possibility that the game will end.

- Now, consider the infinitely repeated Prisoners' Dilemma game, suppose each player's discount factor is δ.
- Suppose, each player's payoffs in this game is the PV of the player's payoffs from the stage games.
- We will show that cooperation, that is (R_1, R_2) can occur in every stage or as SPNE outcome of this game, even though only NE in each stage game is non-cooperation.
- Idea is similar to the finite repeated games. We can think of a simple trigger strategy: cooperate until someone deviates, then switch to noncooperation forever after (or the rest of the game)
- But now we have discount factor involved. We will argue that if discount factor is large (patient) enough, then cooperation is NE or SPNE in this game.

- Trigger strategy for each player can be formally written as

  "Play R_i in the first stage. In the t[th] stage, if the outcome of all t - 1 stages has been (R_1, R_2), then play R_i, otherwise, play L_i."

- If both player adopt trigger strategy then the outcome of the infinitely repeated game will be (R_1, R_2) in every stage.
- Next, we will show that
  - (a) the trigger strategy is NE
  - (b) such NE is SPNE (trigger strategy is NE on every subgame)

- First, for trigger strategy to be the best response, we check this condition:

- Payoff from cooperation is higher than or equal to payoff from deviation once and then payoff from being punished forever.
- if player *j* plays L_j in the first stage, then he gets 5 and then 1 for the rest of the game:

$$5+\delta^1.1+\delta^2.1+\cdots=5+\frac{\delta}{1-\delta}$$

- If player *j* plays R_j , then he will get 4 in this stage and so on = $4+\delta^1.4+\delta^2.4+\cdots=\frac{4}{1-\delta}.$

- So that R_j is optimal if and only if

$$\frac{4}{1-\delta} \geq 5 + \frac{\delta}{1-\delta} \Rightarrow \delta \geq \frac{1}{4}.$$

- Thus, if $\delta \geq 1/4$, then given that player 1 employs trigger strategy, player 2's best response is to use the trigger strategy. Given that player 2 uses trigger strategy, player 1's best response is to play trigger strategy.
- So, the trigger strategy is a Nash equilibrium if $\delta \geq 1/4$.

- Next, we show that such NE strategy is subgame-perfect.

- We need to clarify few concepts: subgame, and strategy in a dynamic game, and subgame perfect
- Subgame is a piece of a game. Or what remains to be played at that point.
- By definition of infinitely repeated game, every subgame of an infinitely repeated game is identical to the game as a whole.
- In two-stage prisoners' dilemma game, there are four subgames in the second stage following the four possible first stage outcome.

Strategy in a dynamic game

- In static game, strategy is simply an action.
- In dynamic game, strategy is a complete plan of action in every contingency.
- In repeated game, strategy is action the player will take in each stage, for each possible history of play through the previous stage.

Subgame perfect NE

Definition (Selten 1965):

A Nash equilibrium is subgame-perfect if the players' strategies constitute a Nash equilibrium in every subgame.

- SPNE as a NE refinement: strategies must be NE and in every subgame

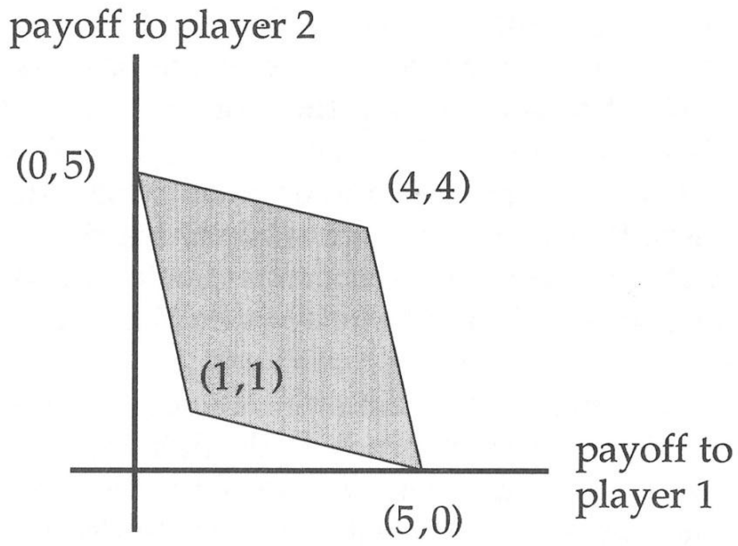
- To show that the trigger strategy Nash equilibrium of the infinitely repeated Prisoners' Dilemma is subgame-perfect, we need to show that trigger strategy is NE on every subgame.
- In trigger strategy Nash equilibrium of infinitely repeated Prisoners' Dilemma, the subgames can be grouped into two classes.
  - (i) subgame in which all earlier outcomes have been (R_1, R_2), and
  - (ii) subgame in which at least one of the earlier outcome differs from (R_1, R_2).

- If players' adopt the trigger strategy for the game as a whole, then players' strategies in a subgame of the first class (i) are the trigger strategy, which is shown to be a Nash equilibrium for the game as a whole.
- As for (ii), players' strategies in the second class of subgame are to repeat (L_1, L_2) forever, which is also a Nash equilibrium of the game as a whole.

Folk Theorem

- If the players are patient enough then any feasible, individually rational payoffs can be enforced to be an equilibrium outcome.
- One-period gain from deviation is outweighed by even a small loss in every future period. That is, you got a minmax in the rest of the game.

- one needs two more definitions.
- The feasible payoffs in the stage game *G* are a convex combination (i.e., a weighted average, where the weights are all nonnegative and sum to one) of the pure-strategy payoffs of *G*.
- The set of feasible payoffs for the Prisoners' Dilemma is the shaded area in Figure 7.



- Next, individually rational payoff for player *i* is the lowest payoff player *i*'s opponents can hold him—minmax value.
- On equilibrium of the repeated game can give any player a payoff lower than his minmax value.
- In Figure 7, (1,1) is minmax value.

- Folk Theorem guarantees that any point in the cross-hatched region in Figure 8 can be achieved as the average payoff in a subgame-perfect Nash equilibrium of the repeated game, provided the discount factor is sufficiently close one.

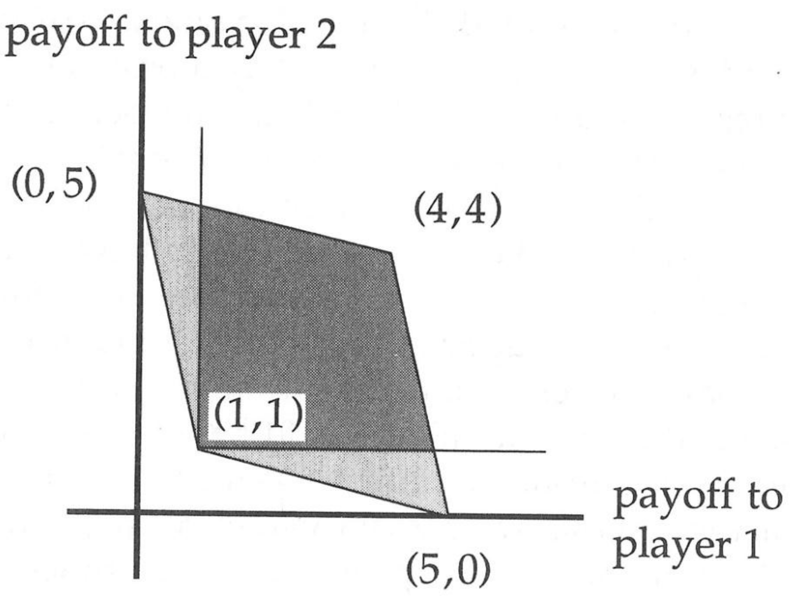


Figure 8

- Payoff in repeated games is PV
- Let define "the payoff that would have to be received in every stage so as to yield the same present value of the whole game" as "the average payoff".
- Suppose the infinite sequence of payoffs has a present value of *V*.
- If the payoff π were received in every stage, the present value would be $\pi/(1-\delta)$.
- For π to be the average payoff from the infinite sequence, these two present values must be equal:

$$V = \pi/(1-\delta) \text{ or } \pi = (1-\delta)V .$$

- Given the discount factor δ, the average payoff of the infinite sequence of payoffs $\pi_1, \pi_2, \pi_3, \ldots$ is $\pi = (1-\delta)\sum_{t=1}^{\infty}\delta^{t-1}\pi_t$
- The advantage of average payoff over present value is that the former is directly comparable to the payoff from stage game.
- Example in prisoners' dilemma, both players get 4 in every period. The average payoff of this infinite sequence is $(1-\delta)(4/(1-\delta)) = 4$.
- But the present value is $4/(1-\delta)$.

Infinitely Repeated Games

- What average payoffs can be achieved by subgame-perfect Nash equilibria when the discount factor is not “sufficiently close to one”?
- The first approach is to consider a fixed value of δ.
- Smaller values of δ make a punishment that will begin next period less effective in deterring a deviation this period.
- Nonetheless, the players typically can do better than simply repeating a stage-game Nash equilibrium.

Infinitely Repeated Games

- Second approach to the problem, pioneered by Abreu (1988), is based on the idea that the most effective way to deter a player from deviating from a proposed strategy is to threaten to administer the strongest credible punishment should the player deviate.
- In most games, switching forever to the stage-game Nash equilibrium is not the strongest credible punishment, so some average payoffs can be achieved using Abreu's approach that cannot be achieved using the trigger-strategy approach.