EE320 (2/2012)

INTRODUCTORY MATHEMATICAL ECONOMICS

MATRIX ALGEBRA AND ITS APPLICATION

(Part 1 – Basic Matrix Algebra)

Topics

- Why do economists need matrix algebra?
- Representation of a system of equations by matrix notations
- Review matrices and matrix operation
  - Types of matrix
  - Matrix operations
- Determinant and singularity of matrix
- Matrix inversion by determinant
- Cramer's rule

Why do economists need matrix algebra?

- Suppose you are asked to solve the following system of simultaneous equations:

$$Q_{d1} = 8 - 3P_1 + P_2 - 4P_3, \quad Q_{s1} = 2 + 2P_1, \quad Q_{d1} = Q_{s1}$$
$$Q_{d2} = 12 + 5P_1 - 2P_2 + 2P_3, \quad Q_{s2} = 2 + 3P_2, \quad Q_{d2} = Q_{s2}$$
$$Q_{d3} = 7 - 6P_1 + P_2 - 3P_3, \quad Q_{s3} = 5 + 4P_3, \quad Q_{d3} = Q_{s3}$$

Question: How would you derive the equilibrium prices and quantities for the 3 goods?

- Linear algebra: solution by elimination of variables

  ➔ This could be quite tedious and prone to mistakes.

- Matrix algebra?

Why do economists need matrix algebra? (Cont'd)

- Matrix algebra can be very useful in handling a large system of simultaneous equations, commonly used in economics.
- Why?
  - It provides a compact way of writing an equation system.
  - It leads to a way of testing the existence of a solution by evaluation of a *determinant*.
  - It gives a method of finding the solution (if exists).
- Drawback: Matrix algebra is applicable only to linear-equation system.

System of Equations in Matrix Form

- Given a system of *m* linear equations in *n* variables:

$$a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n = d_1$$
$$a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n = d_2$$
$$\ldots\ldots\ldots$$
$$a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n = d_m$$

- We can form a matrix from the above system of equations as: $\boldsymbol{AX = d}$, where

$$A = \begin{bmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \ldots & \ldots & \ldots & \ldots \\ a_{m1} & a_{m2} & \ldots & a_{mn} \end{bmatrix} = [a_{ij}] \qquad X = \begin{bmatrix} x_1 \\ x_2 \\ \ldots \\ x_n \end{bmatrix} \qquad d = \begin{bmatrix} d_1 \\ d_2 \\ \ldots \\ d_m \end{bmatrix}$$

Coefficients Variables Constant terms

System of Equations in Matrix Form (Cont'd)

- Example:

$$Q_d = Q_s$$
$$Q_d = a - bP$$
$$Q_s = -c + dP$$

➔Write in matrix form AX = d:

$$A = \begin{bmatrix} 1 & -1 & 0 \\ 1 & 0 & b \\ 0 & 1 & -d \end{bmatrix} \qquad X = \begin{bmatrix} Q_d \\ Q_s \\ P \end{bmatrix} \qquad d = \begin{bmatrix} 0 \\ a \\ -c \end{bmatrix}$$

- Later on we will show that, if $|A| \neq 0$, then $X = A^{-1}d$.

Example

- Suppose a firm sells 3 products in 3 regions.

Region	Sale Volume (Unit: Piece)		
	Good A	Good B	Good C
North	100	80	90
Central	50	110	70
South	120	60	130

- Matrix of sale volumes can be written as:

$$Q = \begin{bmatrix} 100 & 80 & 90 \\ 50 & 110 & 70 \\ 120 & 60 & 130 \end{bmatrix}$$

Types of Matrices

- Let A be an m x n matrix, where m is the number of rows and n is the number of columns.

1. When $m=n$, matrix A is a square matrix.
2. When $n=1$, matrix A is a column vector.
3. When $m=1$, matrix A is a row vector.
4. An identity matrix is a square matrix containing ones along the diagonal and zeros elsewhere:
$$I_n = \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{bmatrix}$$
5. A null matrix is a matrix whose elements are all zero.
6. A diagonal matrix is a square matrix $A_{nxn} = [a_{ij}]$ with $a_{ij}=0$ when $i \neq j$.

Basic Matrix Operations

- Let A and B be $m \times n$ matrices.

1. $A = B$ if and only if $a_{ij} = b_{ij}$ for all values of i and j.
2. If $A = [a_{ij}]_{m \times n}$ and $B = [b_{ij}]_{m \times n}$, then

$$A + B = [a_{ij}]_{m \times n} + [b_{ij}]_{m \times n} = [a_{ij} + b_{ij}]_{m \times n}$$

$$A - B = [a_{ij}]_{m \times n} - [b_{ij}]_{m \times n} = [a_{ij} - b_{ij}]_{m \times n}$$

Example:

$$\begin{bmatrix} 3 & 1 \\ 4 & 7 \end{bmatrix} + \begin{bmatrix} -5 & 2 \\ 6 & 0 \end{bmatrix} = \begin{bmatrix} -2 & 3 \\ 10 & 7 \end{bmatrix}$$

3. If α is a real number, then $\alpha A = \alpha[a_{ij}]_{m \times n} = [\alpha a_{ij}]_{m \times n}$

Example:

$$5\begin{bmatrix} 3 & 1 \\ 4 & 7 \end{bmatrix} = \begin{bmatrix} 15 & 5 \\ 20 & 35 \end{bmatrix}$$

Basic Matrix Operations (cont'd)

4. Rules of Matrix addition and multiplication by scalars
   a) $(A + B) + C = A + (B + C)$
   b) $A + B = B + A$
   c) $A + 0 = A$
   d) $A + (-A) = 0$
   e) $(\alpha + \beta)A = \alpha A + \beta A$
   f) $\alpha (A + B) = \alpha A + \alpha B$

Matrix Multiplication

- Suppose $A = [a_{ij}]_{m\ x\ n}$ and $B = [b_{ij}]_{n\ x\ p}$.

Then, the product $C = AB$ is the $n\ x\ p$ matrix $\mathbf{C = [c_{ij}]_{n\ x\ p}}$, whose element ij is the inner product

$$\boxed{C_{ij} = \sum_{r=1}^{n} a_{ir} b_{rj} = a_{i1} b_{1j} + a_{i2} b_{2j} + \ldots + a_{in} b_{nj}}\ .$$

- Example:

Let $A_{1x2} = \begin{bmatrix} a_{11} & a_{12} \end{bmatrix}$ $\quad B_{2x3} = \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \end{bmatrix}$

$$\Rightarrow C = AB = \begin{bmatrix} a_{11}b_{11} + a_{12}b_{21} & a_{11}b_{12} + a_{12}b_{22} & a_{11}b_{13} + a_{12}b_{23} \end{bmatrix}$$

Example: Matrix Multiplication

- Suppose the sale volume matrix and the price vector are given by:

$$Q = \begin{bmatrix} 100 & 80 & 90 \\ 50 & 110 & 70 \\ 120 & 60 & 130 \end{bmatrix} \qquad P = \begin{bmatrix} P_A \\ P_B \\ P_C \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ 5 \end{bmatrix}$$

- The total revenue from the sales of all products is:

$$TR = QP = \begin{bmatrix} 100 & 80 & 90 \\ 50 & 110 & 70 \\ 120 & 60 & 130 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \\ 5 \end{bmatrix} = \begin{bmatrix} 2(100)+3(80)+5(90) \\ 2(50)+3(110)+5(70) \\ 2(120)+3(60)+5(130) \end{bmatrix} = \begin{bmatrix} 890 \\ 780 \\ 1070 \end{bmatrix}$$

Rule for Matrix Multiplication:

a) $AB \neq BA$ (in general)

b) $(AB)C = A(BC)$

c) $(A + B)C = AB + BC$

d) $(B + C)A = BA + CA$

- Example:

Let $A = \begin{bmatrix} 3 & 1 \\ 4 & 7 \end{bmatrix}$ $B = \begin{bmatrix} -5 & 2 \\ 6 & 0 \end{bmatrix}$

→ $AB = \begin{bmatrix} 3 & 1 \\ 4 & 7 \end{bmatrix}\begin{bmatrix} -5 & 2 \\ 6 & 0 \end{bmatrix} = \begin{bmatrix} -15+6 & 6 \\ -20+42 & 8 \end{bmatrix} = \begin{bmatrix} -9 & 6 \\ 22 & 8 \end{bmatrix}$

→ $BA = \begin{bmatrix} -5 & 2 \\ 6 & 0 \end{bmatrix}\begin{bmatrix} 3 & 1 \\ 4 & 7 \end{bmatrix} = \begin{bmatrix} -15+8 & -5+14 \\ 18 & 6 \end{bmatrix} = \begin{bmatrix} -7 & 9 \\ 18 & 6 \end{bmatrix}$

Matrix Transposition

- The transpose of any *m x n* matrix *A*, denoted by *A′* or A^T, is defined as the *n x m* matrix whose first column is the first row of *A*, and the second column is the second row of *A*, and so on.

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix} \longrightarrow A' = \begin{bmatrix} a_{11} & a_{21} & \dots & a_{m1} \\ a_{12} & a_{22} & \dots & a_{m2} \\ \dots & \dots & \dots & \dots \\ a_{1n} & a_{2n} & \dots & a_{mn} \end{bmatrix}$$

- Example:

$$Q = \begin{bmatrix} 100 & 80 & 90 \\ 50 & 110 & 70 \\ 120 & 60 & 130 \end{bmatrix} \longrightarrow Q' = \begin{bmatrix} 100 & 50 & 120 \\ 80 & 110 & 60 \\ 90 & 60 & 130 \end{bmatrix}$$

Properties of transposes

a) $(A')' = A$

b) $(A + B)' = A' + B'$

c) $(AB)' = B'A'$

d) $(\alpha A)' = \alpha A'$

Note: A square matrix A is symmetric (i.e. $A = A'$) iff $a_{ij} = a_{ji}$ for all i,j

- Example:

$$A = \begin{bmatrix} 3 & 1 \\ 4 & 7 \end{bmatrix} \qquad B = \begin{bmatrix} -5 & 2 \\ 6 & 0 \end{bmatrix}$$

→ $$(AB)' = \begin{bmatrix} -9 & 6 \\ 22 & 8 \end{bmatrix}' = \begin{bmatrix} -9 & 22 \\ 6 & 8 \end{bmatrix}$$

→ $$B'A' = \begin{bmatrix} -5 & 6 \\ 2 & 0 \end{bmatrix}\begin{bmatrix} 3 & 4 \\ 1 & 7 \end{bmatrix} = \begin{bmatrix} -15+6 & -20+42 \\ 6 & 8 \end{bmatrix} = \begin{bmatrix} -9 & 22 \\ 6 & 8 \end{bmatrix}$$

Matrix Inversion

- The inverse of matrix A, denoted by A^{-1}, is defined only if A is a square matrix and must satisfy the condition

$$AA^{-1} = A^{-1}A = I.$$

- Properties of the inverse:
  a) If A^{-1} exists, then $(A^{-1})^{-1} = A$.
  b) If AB is invertible, then $(AB)^{-1} = B^{-1}A^{-1}$
  c) $(A')^{-1} = (A^{-1})'$
  d) $(cA)^{-1} = c^{-1}A^{-1}$, where $c \neq 0$
- If a square matrix A has an inverse, A is said to be nonsingular. If A is not invertible, then A is called a singular matrix.
  - The nonsingularity condition is required for a solution of a linear-equation system to exist.

Inverse Matrix and Solution of Linear-Equation System

- Given the equation system in matrix notation:

$$AX = d$$

$$A^{-1}AX = A^{-1}d$$

$$\Rightarrow \quad X = A^{-1}d$$

where X is the column vector of variables, and $A^{-1}d$ is the column vector of solution values.

  - Since A^{-1}, if it exists, is unique, the solution $A^{-1}d$ must be unique values.

- Example:

$$\begin{bmatrix} Q_d \\ Q_s \\ P \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 \\ 1 & 0 & b \\ 0 & 1 & -d \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ -a \\ c \end{bmatrix} = \frac{-1}{b+d} \begin{bmatrix} -b & -d & -b \\ d & -d & -b \\ 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ -a \\ c \end{bmatrix} = \frac{1}{b+d} \begin{bmatrix} ad-bc \\ ad-bc \\ a+c \end{bmatrix}$$

- Next, we will study how to determine A^{-1} by determinant.

Determinant

- One way to test whether an inverse of a matrix exists (i.e. it is nonsingular) is to use determinant.
- The determinant of a *square* matrix A, denoted by |A|, is a uniquely defined associated with that matrix.
  - Determinant of order 1: |A| = |a| = a
  - Determinant of order 2: $|A| = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{21}a_{12}$
  - Determinant of order 3:

$$|A| = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

➢ $$|A| = a_{11}\left(a_{22}a_{33} - a_{32}a_{23}\right) - a_{12}\left(a_{21}a_{33} - a_{31}a_{23}\right) + a_{13}\left(a_{21}a_{32} - a_{31}a_{22}\right)$$

Example: Determinants of Order 3

- Evaluate the following determinant:

$$|A| = \begin{vmatrix} 5 & 7 & 9 \\ 2 & 5 & 6 \\ 9 & 0 & 12 \end{vmatrix}$$

$$= 5\begin{vmatrix} 5 & 6 \\ 0 & 12 \end{vmatrix} - 7\begin{vmatrix} 2 & 6 \\ 9 & 12 \end{vmatrix} + 9\begin{vmatrix} 2 & 5 \\ 9 & 0 \end{vmatrix}$$

$$= 5(60 - 0) - 7(24 - 54) + 9(0 - 45)$$

$$= 300 + 210 - 405$$

$$= 105$$

Evaluating an *n*th-Order Determinants by Laplace Expansion

Definitions: Let A be an n x n matrix.

- The minor of the element a_{ij} is: $|M_{ij}| = \begin{vmatrix} a_{11} & \cdots & a_{1j-1} & a_{1j} & a_{1j+1} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots & \vdots & \vdots & \iddots & \vdots \\ a_{i1} & \cdots & a_{ij-1} & a_{ij} & a_{ij+1} & \cdots & a_{in} \\ \vdots & \iddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nj-1} & a_{nj} & a_{nj+1} & \cdots & a_{nn} \end{vmatrix}$

- The cofactor of the element a_{ij} is: $|C_{ij}| = (-1)^{i+j}\,|M_{ij}|$

- The determinant of the matrix A of order n can be found by the Laplace expansion of *any row* or *any column* as follows:

$$|A| = \sum_{j=1}^{n} a_{ij}|C_{ij}| \quad \text{[expansion by the } i\text{th row]}$$

$$|A| = \sum_{i=1}^{n} a_{ij}|C_{ij}| \quad \text{[expansion by the } j\text{th column]}$$

Basic Properties of Determinants (1)

- <u>Property I</u>

The interchange of rows and columns does not affect the value of a determinant. That is, |A'| = |A|.

Example:

$$|A| = \begin{vmatrix} 3 & 1 \\ 4 & 7 \end{vmatrix} = 17 \qquad |A'| = \begin{vmatrix} 3 & 4 \\ 1 & 7 \end{vmatrix} = 17$$

- <u>Property II</u>

The interchange of any two rows (or columns) will alter the sign, but not the numerical value of the determinant.

Example:

$$|A| = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc \qquad |B| = \begin{vmatrix} c & d \\ a & b \end{vmatrix} = bc - ad$$

Basic Properties of Determinants (2)

- <u>Property III</u>

If all element in any one row (or column) is multiplied by a scalar k, the determinant is multiplied by k.

Example:

$$|A| = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc \qquad |B| = \begin{vmatrix} a & b \\ kc & kd \end{vmatrix} = kad - kbc = k|A|$$

- <u>Property IV</u>

The addition (or subtraction) of a multiple of any row to (from) another row will leave the value of the determinant unaltered.

Example:

$$|A| = \begin{vmatrix} a & b \\ c & d \end{vmatrix} \qquad |B| = \begin{vmatrix} a & b \\ c+ka & d+kb \end{vmatrix} = ad+kab - bc - kab$$

$$= |A|$$

Basic Properties of Determinants (3)

- <u>Property V</u>

If one row (or column) is a multiple of another row (or column), the value of the determinant will be zero.

Example:

$$[A]=\begin{bmatrix} 3 & 4 & 2 \\ 15 & 20 & 10 \\ 4 & 0 & 1 \end{bmatrix}$$

➔ |A| = 0. Thus, this matrix A is linearly dependent.

<u>Note</u>:

The condition of *linear independence* is a sufficient condition for the *nonsingularity* of a matrix. For the rows (or columns) to be <u>linearly independent</u>, none must be a linear combination of the rest.

Determinantal Criterion for Nonsingularity

- Summary:

$|A| \neq 0 \Leftrightarrow$ there is row (column) independence in matrix A

A is nonsingular

A^{-1} exists

a unique solution $X^* = A^{-1}d$ exists

Matrix Inversion by Determinant

- Assume that A is an n x n nonsingular matrix, and |A|≠ 0.
- The inverse of matrix A is:

$$\boxed{A^{-1} = \frac{1}{|A|} adj(A)}$$

where

$$adj(A) \equiv C'_{nxn} \equiv \begin{bmatrix} |C_{11}| & |C_{21}| & \cdots & |C_{n1}| \\ |C_{12}| & |C_{22}| & \cdots & |C_{n2}| \\ \cdots & \cdots & \cdots & \cdots \\ |C_{1n}| & |C_{2n}| & \cdots & |C_{nn}| \end{bmatrix}$$

and

$$|C_{ij}| = (-1)^{i+j} \begin{vmatrix} a_{11} & \cdots & a_{1j-1} & a_{1j} & a_{1j+1} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots & \vdots & \vdots & \iddots & \vdots \\ a_{i1} & \cdots & a_{ij-1} & a_{ij} & a_{ij+1} & \cdots & a_{in} \\ \vdots & \iddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nj-1} & a_{nj} & a_{nj+1} & \cdots & a_{nn} \end{vmatrix}$$

Steps to Find the Inverse of a Matrix A

1. Find the determinant $|A|$. If $|A| = 0$, then the inverse does not exist.
2. Find the cofactors of all the elements of A, and arrange them as a matrix: $C = [|C_{ij}|]$
3. Take the transpose of the cofactor matrix, C, to get the *adj(A)*.
4. Divide the adj(A) by the determinant.

Example:

Find the inverse of $A = \begin{bmatrix} 1 & -1 & 0 \\ 1 & 0 & b \\ 0 & 1 & -d \end{bmatrix}$

Answer:

$$A^{-1} = \frac{1}{-(b+d)} \begin{bmatrix} -b & -d & -b \\ d & -d & -b \\ 1 & -1 & 1 \end{bmatrix}$$

Finding Solutions Using Inverse Matrix

- For a problem AX = d, if |A| ≠ 0, then X = A^{-1}d.
- Example:

$$A = \begin{bmatrix} 1 & -1 & 0 \\ 1 & 0 & b \\ 0 & 1 & -d \end{bmatrix} \qquad X = \begin{bmatrix} Q_d \\ Q_s \\ P \end{bmatrix} \qquad d = \begin{bmatrix} 0 \\ a \\ -c \end{bmatrix}$$

- Solution:

$$X^* = \begin{bmatrix} Q_d^* \\ Q_s^* \\ P^* \end{bmatrix} = A^{-1}d = \frac{1}{-(b+d)} \begin{bmatrix} -b & -d & -b \\ d & -d & -b \\ 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ a \\ -c \end{bmatrix} = \frac{1}{b+d} \begin{bmatrix} ad-bc \\ ad-bc \\ a+c \end{bmatrix}$$

Cramer's Rule

- Given an equation system *Ax = d*, where *A* is an *n x n* nonsingular matrix, the solution value of the *j*th variable can be obtained from:

$$x_j^* = \frac{|A_j|}{|A|} = \frac{1}{|A|}\begin{vmatrix} a_{11} & a_{12} & \cdots & \boxed{d_1} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & d_2 & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & d_n & \cdots & a_{nn} \end{vmatrix}$$

where $|A_j|$ is the determinant of the matrix *A* when the *j*th column is replaced by the constant terms $d_1 \dots d_n$.

Example: Cramer's Rule

- Given the system of two equations:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} u \\ v \end{bmatrix}$$

- By Cramer's rule, the solutions are given by:

$$x^* = \frac{\begin{vmatrix} u & b \\ v & d \end{vmatrix}}{\begin{vmatrix} a & b \\ c & d \end{vmatrix}} = \frac{ud - bv}{ad - bc} \quad \text{and} \quad y^* = \frac{\begin{vmatrix} a & u \\ c & v \end{vmatrix}}{\begin{vmatrix} a & b \\ c & d \end{vmatrix}} = \frac{av - uc}{ad - bc}$$

Another Example: Market Equilibrium

$$A=\begin{bmatrix}1 & -1 & 0\\ 1 & 0 & b\\ 0 & 1 & -d\end{bmatrix} \qquad X=\begin{bmatrix}Q_d\\ Q_s\\ P\end{bmatrix} \qquad d=\begin{bmatrix}0\\ a\\ -c\end{bmatrix}$$

- Using Cramer's rule, we get:

$$Q^*=\frac{\begin{vmatrix}0 & -1 & 0\\ a & 0 & b\\ -c & 1 & -d\end{vmatrix}}{\begin{vmatrix}1 & -1 & 0\\ 1 & 0 & b\\ 0 & 1 & -d\end{vmatrix}}=\frac{ad-bc}{b+d} \quad \text{and} \quad P^*=\frac{\begin{vmatrix}1 & -1 & 0\\ 1 & 0 & a\\ 0 & 1 & -c\end{vmatrix}}{\begin{vmatrix}1 & -1 & 0\\ 1 & 0 & b\\ 0 & 1 & -d\end{vmatrix}}=\frac{a+c}{b+d}$$