Assignment 3: Due date: March 31, 2022 before 2.00 pm

On page 134-138, Chapter 4: Consumption-savings and state pricing, work on the following questions:

$$2, 3, 4, 6$$

2. Individual utility function: $\ln(c_0) + E[\delta \ln(c_1)]$

$$P_i = E\left[\frac{\delta U'(c_1)}{U'(c_0)} x_i\right] = E[m_{01} x_i]$$

$$P_s = P' X^{-1} e_s$$

3. max $U(c_0) + \delta E[U(\tilde{c}_1)]$ subject to $\sum_{i=1}^{n} w_i = 1$ is

$$\mathcal{L} = U(c_0) + \delta E\left[U\left[y_1 + (w_0 + y_0 - c_0)\sum_{i=1}^{n} w_i R_i\right]\right] + \lambda'\left[1 - \sum_{i=1}^{n} w_i\right]$$

$$\frac{\partial \mathcal{L}}{\partial c_0} = U'(c_0) + \delta E\left[U'(c_1)(-1)\sum_{i=1}^{n} w_i R_i\right] = 0$$

$$U'(c_0) = \delta E\left[U'(c_1)\cancel{\sum_{i=1}^{n}} w_i R_i\right] \qquad \text{given } \lambda = \delta E[U'(c_1) R_i]$$

$$= \lambda$$

$$\therefore U'(c_0) = \delta E[U'(c_1) R_i]$$

Equation for an elementary security s that pays R_s is U

$$U'(c_0) = R_s \delta E[U'(c_s)]$$

$$\frac{1}{R_s} = \delta E\left[\left(\frac{c_s}{c_1}\right)^{1-\gamma}\right] \qquad U'(c) = \frac{\gamma c^{\gamma-1}}{\gamma} = c^{\gamma-1}$$

in real world ; $R_s = \frac{1}{\delta}\left(\frac{c_s}{c_0}\right)^{1-\gamma}$

$$\frac{\partial R_s}{\partial c_s/c_0} = \frac{1-\gamma}{\delta}\left(\frac{c_s}{c_0}\right)^{-\gamma}$$

$$\delta = \frac{\left(\frac{c_s}{c_0}\right)^{1-\gamma}}{R_s} \quad ; \quad = \frac{1-\gamma (R_s)\left(\frac{c_s}{c_0}\right)^{-\gamma}}{\left(\frac{c_s}{c_0}\right)^{1-\gamma}}$$

$$\frac{\partial R_s}{\partial c_s/c_0} = (1-\gamma)\frac{R_s}{\frac{c_s}{c_0}}$$

so that $\epsilon = \frac{R_s \partial \frac{c_s}{c_0}}{\frac{c_s}{c_0} \partial R_s} = \frac{1}{1-\gamma}$

4. $r_f = 5\%$ $R_f = 1.05$ $X = \begin{bmatrix} 1 & 10 \\ 1 & 5 \end{bmatrix}$ $p = \begin{bmatrix} 0.95 \\ 6 \end{bmatrix}$ $|X| = -5$ $adj(X) = \begin{bmatrix} +5 & -1 \\ -10 & +1 \end{bmatrix}^T = \begin{bmatrix} 5 & -10 \\ -1 & 1 \end{bmatrix}$ $X^{-1} = \begin{bmatrix} -1 & 2 \\ \frac{1}{5} & \frac{-1}{5} \end{bmatrix}$

a) $p_1 = p'X^{-1}e_1 = [0.95 \;\; 6]\begin{bmatrix} -1 & 2 \\ 1/5 & -1/5 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = 0.2476$

$p_2 = p'X^{-1}e_2 = [0.95 \;\; 6]\begin{bmatrix} -1 & 2 \\ 1/5 & -1/5 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = 0.7047$

$\hat{\pi}_1 = P_1 R_f = 0.2476(1.05) = 0.26$ and $\hat{\pi}_2 = P_2 R_f = 0.7047(1.05) = 0.74$

b) $m_1 = \dfrac{0.2476}{0.5} = 0.4952$

$m_2 = \dfrac{0.7048}{0.5} = 1.4096$

6. a) $E[a + bR_m(R_i - R_f)] = 0$

$E[aR_i - aR_f + bR_mR_i - bR_mR_f] = 0$

$aE(R_i) - aR_f + bE(R_mR_i) - bR_fE(R_m) = 0$ $E(R_mR_i) = E(R_m)E(R_i) + COV(R_m, R_i)$

$a[E(R_i) - R_f] + b[E(R_m)E(R_i) + COV(R_m, R_i) - R_fE(R_m)] = 0$

$a[E(R_i) - R_f] + b[E(R_m)(E(R_i) - R_f) + COV(R_m, R_i)] = 0$

$[E(R_i) - R_f][a + bE(R_m)] + b\,cov(R_m, R_i) = 0$

given $E(R_i) = R_f + \beta_i\gamma$; $\beta_i\gamma[a + bE(R_m)] + b\,cov(R_m, R_i) = 0$

$E(R_i) - R_f = \beta_i\gamma$

$\gamma = \dfrac{-b\,cov(R_i, R_m)}{\beta_i[a + bE(R_m)]}$ $\beta_i = \dfrac{cov(i,m)}{\sigma_m^2}$

γ: in $a, b, E(R_m), \sigma_m^2$ $\therefore$ $\gamma = \dfrac{-b\sigma_m^2}{a + bE(R_m)}$

b) $\dfrac{1}{R_f} = E(a + bR_m)$

$= a + bE(R_m)$ risk free $\longrightarrow$ $a = \dfrac{1}{R_f} - bE(R_m)$

$1 = E[(a + bR_m)R_m]$

$= aE(R_m) + bE(R_m^2)$ portfolio

find b; $1 = \left[\dfrac{1}{R_f} - bE(R_m)\right]E(R_m) + bE(R_m^2)$

$= \dfrac{E(R_m)}{R_f} - bE(R_m)^2 + bE(R_m^2)$

$= \dfrac{E(R_m)}{R_f} + b\underbrace{[E(R_m^2) - E(R_m)^2]}_{\sigma_m^2}$

$1 = \dfrac{E(R_m)}{R_f} + b\sigma_m^2$

$\therefore b = \dfrac{1 - \frac{E(R_m)}{R_f}}{\sigma_m^2} = \dfrac{R_f - E(R_m)}{R_f\sigma_m^2}$

$$\therefore \quad a = \frac{1}{R_f} - \left[\frac{R_f - E(R_m)}{R_f \sigma_m^2}\right] \cdot E(R_m)$$

c) from (a) $\quad \gamma = \dfrac{-b\sigma_m^2}{a + bE(R_m)}$

$$= \frac{-\left[\dfrac{R_f - E(R_m)}{R_f\sigma_m^2}\right]\sigma_m^2}{\dfrac{1}{R_f} - \left[\dfrac{R_f - E(R_m)}{R_f\sigma_m^2}\right] \cdot E(R_m) + \left[\dfrac{R_f - E(R_m)}{R_f\sigma_m^2}\right] E(R_m)}$$

$$= \frac{\dfrac{[E(R_m) - R_f]\cancel{\sigma_m^2}}{R_f\cancel{\sigma_m^2}}}{\dfrac{\sigma_m^2 - \cancel{E(R_m)[R_f - E(R_m)]} + \cancel{[R_f - E(R_m)]E(R_m)}}{R_f\sigma_m^2}}$$

$$= \frac{E(R_m) - R_f}{\cancel{R_f}}\left(\frac{\cancel{R_f}\cancel{\sigma_m^2}}{\cancel{\sigma_m^2}}\right)$$

$$\therefore \quad \gamma = E(R_m) - R_f$$