Chapter 2: Determinants

In this chapter we will study "determinants" or, more precisely,"determinant functions," which assigns a real number $\det(\mathbf{A})$ to a matrix variable $\mathbf{A}$. Although determinants first arose in the context of solving systems of linear equations, they are rarely used for that purpose in real-world applications. While they can be useful for solving very small linear systems (say two or three unknowns), our main interest in them stems from the fact that they link together various concepts in linear algebra and provide a useful formula for the inverse of a matrix.

1 Determinants by Cofactor Expansion

There are various methods for defining determinants of higher-order square matrices. In this text, we will us an "inductive definition" by which we mean that the determinant of a square matrix of a given order will be defined in terms of determinants of square matrices of the next lower order.

- For $\mathbf{A} = [a] \in \mathbb{R}$,we define
$$\det(\mathbf{A}) = \det[a] = a.$$
- For $\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \in \mathbb{R}^{2\times 2}$
$$\det(\mathbf{A}) = a_{11}a_{22} - a_{12}a_{21} = \det[a_{11}]\det[a_{22}] - \det[a_{12}]\det[a_{21}].$$
- We can define determinants of 3×3 matrices in terms of determinants of 2×2 matrices, then determinants of 4×4 matrices in terms of determinants of 3×3 matrices, and so forth. The following terminology and notation will help to make this inductive process more efficient.

Definition 1.1. If $\mathbf{A}$ is a square matrix, then the **minor** of entry a_{ij} is denoted by $\mathbf{M}_{ij}$ and is defined to be the determinant of the submatrix that remains after the i-th row and j-th column are deleted from $\mathbf{A}$.

The **cofactor** of entry a_{ij}, denoted by $\mathbf{C}_{ij}$, is defined by
$$\mathbf{C}_{ij} = (-1)^{i+j}\mathbf{M}_{ij}.$$

Example 1.1. Consider $\mathbf{A} = \begin{bmatrix} 3 & 1 & -4 \\ 2 & 5 & 6 \\ 1 & 4 & 8 \end{bmatrix}$. Find minors $\mathbf{M}_{11}, \mathbf{M}_{32}$ and cofactors $\mathbf{C}_{11}, \mathbf{C}_{32}$

Remark: A minor M_{ij} and its corresponding cofactor C_{ij} are either the same or negatives of each other and that the relating sign $(1)^{i+j}$ is either $+1$or -1 in accordance with the pattern in the checkerboard array.

$$\begin{bmatrix} + & - & + & - & + & \cdots \\ - & + & - & + & - & \cdots \\ + & - & + & - & + & \cdots \\ - & + & - & + & - & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \end{bmatrix}.$$

Example 1.2. Cofactor Expansions of a 2×2 Matrix
The checkerboard pattern for a 2×2 matrix $\mathbf{A} = [a_{ij}]$ is

$$\begin{bmatrix} + & - \\ - & + \end{bmatrix}$$

$$C_{11} = M_{11} = a_{22} \qquad C_{12} = -M_{12} = -a_{21}$$
$$C_{21} = -M_{21} = -a_{12} \qquad C_{22} = M_{22} = a_{11}$$

From the formula $\det(\mathbf{A}) = a_{11}a_{22} - a_{12}a_{21}$.
it can be shown that

$$\begin{aligned} \det(A) &= \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \\ &= a_{11}C_{11} + a_{12}C_{12} \\ &= a_{21}C_{21} + a_{22}C_{22} \\ &= a_{11}C_{11} + a_{21}C_{21} \\ &= a_{12}C_{12} + a_{22}C_{22} \end{aligned}$$

Remarks:
- Each of the last four equations is called a **cofactor expansion** of $\det(\mathbf{A})$.
- In each cofactor expansion, the entries and cofactors all come from the same row or same column of $\mathbf{A}$.

Theorem 1.1. If A is an $n \times n$ matrix, then regardless of which row or column of **A** is chosen, the number obtained by multiplying the entries in that row or column by the corresponding cofactors and adding the resulting products is always the same.

This result allows us to make the following definition of **determinant**.

Definition 1.2. If **A** is an $n \times n$ matrix, then the number obtained by multiplying the entries in any row or column of **A** by the corresponding cofactors and adding the resulting products is called the **determinant** of **A**, and the sums themselves are called **cofactor expansions** of A.

- The determinant obtined from "cofactor expansion along the j-th column" is given by

$$\det(A) = a_{1j}C_{1j} + a_{2j}C_{2j} + ... + a_{nj}C_{nj}.$$

- The determinant obtined from "cofactor expansion along the i-th row" is given by

$$\det(A) = a_{i1}C_{i1} + a_{i2}C_{i2} + ... + a_{in}C_{in}.$$

Example 1.3. Consider the matrix $\mathbf{A} = \begin{bmatrix} 3 & 1 & 0 \\ -2 & -4 & 3 \\ 5 & 4 & -2 \end{bmatrix}$.

Find the determinant of the matrix by
(i) cofactor expansion along the first row of **A**, and
(ii)cofactor expansion along the first column of **A**.

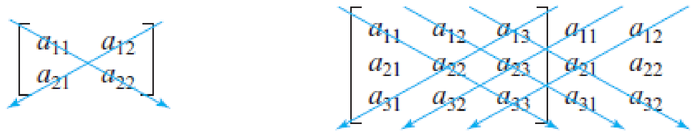
Example 1.4 (Smart Choice of Row or Column)**.** Let $\mathbf{A} = \begin{bmatrix} 1 & 0 & 0 & -1 \\ 3 & 1 & 2 & 2 \\ 1 & 0 & -2 & 1 \\ 2 & 0 & 0 & 1 \end{bmatrix}$. Find $\det(\mathbf{A})$.

Example 1.5 (Determinant of Triangular Matrix)**.** Let $\mathbf{A} = \begin{bmatrix} a_{11} & 0 & 0 & 0 \\ a_{21} & a_{22} & 0 & 0 \\ a_{31} & a_{32} & a_{33} & 1 \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix}$. Find $\det(\mathbf{A})$.

Remark: This example shows that the determinant of a lower triangular matrix is the product of its diagonal entries. Each part of the computation uses a cofactor expansion along the first row.

Theorem 1.2. If $\mathbf{A}$ is an $n \times n$ triangular matrix (upper triangular, lower triangular, or diagonal), then $\det(\mathbf{A})$ is the product of the entries on the main diagonal of the matrix; that is, $\boxed{\det(\mathbf{A}) = a_{11}a_{22}\ldots a_{nn}}$.

A UsefulTechnique for Evaluating 2×2 and 3×3 Determinants



$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}$$

$$\begin{aligned}\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} &= a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} \\ &= a_{11}(a_{22}a_{33} - a_{23}a_{32}) - a_{12}(a_{21}a_{33} - a_{23}a_{31}) + a_{13}(a_{21}a_{32} - a_{22}a_{31}) \\ &= a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31} - a_{12}a_{21}a_{33} - a_{11}a_{23}a_{32}\end{aligned}$$

- In the 2×2 case, the determinant can be computed by forming the product of the entries on the rightward arrow and subtracting the product of the entries on the leftward arrow.
- In the 3×3 case we first recopy the first and second columns as shown in the figure, after which we can compute the determinant by summing the products of the entries on the rightward arrows and subtracting the products on the leftward arrows.

Example 1.6. From the technique given above, we can compute determinants as follows.

$$\begin{vmatrix} 3 & 1 \\ 4 & -2 \end{vmatrix} = \begin{vmatrix} 3 & 1 \\ 4 & -2 \end{vmatrix} = (3)(-2) - (1)(4) = -10$$

$$\begin{vmatrix} 1 & 2 & 3 \\ -4 & 5 & 6 \\ 7 & -8 & 9 \end{vmatrix} = \left|\begin{matrix} 1 & 2 & 3 \\ -4 & 5 & 6 \\ 7 & -8 & 9 \end{matrix}\right| \begin{matrix} 1 & 2 \\ -4 & 5 \\ 7 & -8 \end{matrix}$$

$$= [45 + 84 + 96] - [105 - 48 - 72] = 240$$

2 Evaluating Determinants by Row Reduction

In this section we will show how to evaluate a determinant by reducing the associated matrix to row echelon form. In general, this method requires less computation than cofactor expansion and hence is the method of choice for large matrices.

Theorem 2.1. Let **A** be a square matrix. If **A** has a row of zeros or a column of zeros, then $\det(A) = 0$.

Proof: Since the determinant of **A** can be found by a cofactor expansion along any row or column, we can use the row or column of zeros. Thus, if we let $\mathbf{C}_1, \mathbf{C}_2, \ldots, \mathbf{C}_n$ denote the cofactors of **A** along that row or column, then

$$\det(A) = 0 \cdot \mathbf{C}_1 + 0 \cdot \mathbf{C}_2 + \cdots + 0 \cdot \mathbf{C}_n = 0$$

■

Theorem 2.2. Let **A** be a square matrix. Then $\det(\mathbf{A}) = \det(\mathbf{A}^T)$.

Proof: Since transposing a matrix changes its columns to rows and its rows to columns,the cofactor expansion of **A** along any row is the same as the cofactor expansion of $\mathbf{A}^T$ along the corresponding column. Thus, both have the same determinant.

Relationship	Operation
$\begin{vmatrix} ka_{11} & ka_{12} & ka_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = k \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$ $\det(B) = k\det(A)$	In the matrix B the first row of A was multiplied by k.
$\begin{vmatrix} a_{21} & a_{22} & a_{23} \\ a_{11} & a_{12} & a_{13} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = - \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$ $\det(B) = -\det(A)$	In the matrix B the first and second rows of A were interchanged.
$\begin{vmatrix} a_{11}+ka_{21} & a_{12}+ka_{22} & a_{13}+ka_{23} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$ $\det(B) = \det(A)$	In the matrix B a multiple of the second row of A was added to the first row.

Theorem 2.3. Let $\mathbf{A}$ be an $n \times n$ matrix.

(a) If $\mathbf{B}$ is the matrix that results when a single row or single column of $\mathbf{A}$ is multiplied by a scalar k, then $\det(\mathbf{B}) = k \det(\mathbf{A})$.

(b) If $\mathbf{B}$ is the matrix that results when two rows or two columns of $\mathbf{A}$ are interchanged, then $\det(\mathbf{B}) = -\det(\mathbf{A})$.

(c) If $\mathbf{B}$ is the matrix that results when a multiple of one row of $\mathbf{A}$ is added to another or when a multiple of one column is added to another, then $\det(\mathbf{B}) = \det(\mathbf{A})$.

Proof (a):

Theorem 2.4. Let $\mathbf{E}$ be an $n \times n$ elementary matrix.

(a) If $\mathbf{E}$ results from multiplying a row of In by a nonzero number k, then $\det(\mathbf{E}) = k$.

(b) If $\mathbf{E}$ results from interchanging two rows of $\mathbf{I}_n$, then $\det(\mathbf{E}) = -1$.

(c) If $\mathbf{E}$ results from adding a multiple of one row of I_n to another, then $\det(\mathbf{E}) = 1$.

Note: This is a special case of Theorem 2.3 in which $\mathbf{A} = \mathbf{I}_n$ is the $n \times n$ identity matrix and $\mathbf{E}$ denotes the elementary matrix that results when the row operation is performed on I_n. ■

Theorem 2.5. If $\mathbf{A}$ is a square matrix with two proportional rows or two proportional columns, then $\det(\mathbf{A}) = 0$.

Proof: If a square matrix $\mathbf{A}$ has two proportional rows, then a row of zeros can be introduced by adding a suitable multiple of one of the rows to the other. Similarly for columns. But adding a multiple of one row or column to another does not change the determinant, so from Theorem 2.1, we must have $\det(\mathbf{A}) = 0$. This proves the above theorem.

Example 2.1. Each of the following matrices has two proportional rows or columns; thus, each has a determinant of zero.

$$\begin{bmatrix} -1 & 4 \\ -2 & 8 \end{bmatrix}, \quad \begin{bmatrix} 1 & -2 & 7 \\ -4 & 8 & 5 \\ 2 & -4 & 3 \end{bmatrix}, \quad \begin{bmatrix} 3 & -1 & 4 & -5 \\ 6 & -2 & 5 & 2 \\ 5 & 8 & 1 & 4 \\ -9 & 3 & -12 & 15 \end{bmatrix}$$

Example 2.2. Evaluate $\det(\mathbf{A})$ by using row reduction where $\mathbf{A} = \begin{bmatrix} 0 & 1 & 5 \\ 3 & -6 & 9 \\ 2 & 6 & 1 \end{bmatrix}$.

Solution: We will reduce A to row echelon form (which is upper triangular) and then apply Theorem 1.2

Example 2.3. Compute the determinant of $\mathbf{A} = \begin{bmatrix} 1 & 0 & 0 & 3 \\ 2 & 7 & 0 & 6 \\ 0 & 6 & 3 & 0 \\ 7 & 3 & 1 & -5 \end{bmatrix}$.

[Hint: Use ERO on $\mathbf{A}^T$, i.e. using column operations to evaluate determinant]
Solution: We will reduce $\mathbf{A}^T$ to row echelon form (which is upper triangular) and then apply Theorem 1.2

Example 2.4. Compute the determinant of $\mathbf{A} = \begin{bmatrix} 1 & 0 & 0 & 3 \\ 2 & 7 & 0 & 6 \\ 0 & 6 & 3 & 0 \\ 7 & 3 & 1 & -5 \end{bmatrix}$.

[Hint: Use row operations and cofactor expansion]
Solution:

3 Properties of Determinants; Cramers Rule

In this section we will develop some fundamental properties of matrices, and we will use these results to derive a formula for the inverse of an invertible matrix and formulas for the solutions of certain kinds of linear systems

Theorem 3.1. Suppose that $\mathbf{A}$ is an $n \times n$ matrix and k is any scalar. Then

$$\det(k\mathbf{A}) = k^n \det(\mathbf{A}).$$

Proof: Since a common factor of any row of a matrix can be moved through the determinant sign, and since each of the n rows in $k\mathbf{A}$ has a common factor of k, then we have $\det(k\mathbf{A}) = k^n \det(\mathbf{A})$.

Note: In general, $\det(\mathbf{A} + \mathbf{B}) neq \det(\mathbf{A}) + \det(\mathbf{B})$.

Theorem 3.2. Let $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{C}$ be $n \times n$ matrices that differ only in a single row, say the r-th, and assume that the r-th row of $\mathbf{C}$ can be obtained by adding corresponding entries in the r-th rows of $\mathbf{A}$ and $\mathbf{B}$. Then

$$\det(\mathbf{C}) = \det(\mathbf{A}) + \det(\mathbf{B})$$

The same result holds for columns.

Case n = 2:

Theorem 3.3. If $\mathbf{B}$ is an $n \times n$ matrix and $\mathbf{E}$ is an $n \times n$ elementary matrix, then

$$\det(\mathbf{EB}) = \det(\mathbf{E})\det(\mathbf{B}).$$

It follows that by repeated applications of above theorem that if $\mathbf{B}$ is an $n \times n$ matrix and $\mathbf{E}_1, \mathbf{E}_2, \ldots, \mathbf{E}_r$ are $n \times n$ elementary matrices, then

$$\det(\mathbf{E}_1\mathbf{E}_2 \cdots \mathbf{E}_r\mathbf{B}) = \det(\mathbf{E}_1) det(\mathbf{E}_2) \cdots \det(\mathbf{E}_r)\det(\mathbf{B}) \tag{1}$$

Theorem 3.4. A square matrix $\mathbf{A}$ is invertible if and only if $\det(\mathbf{A}) \neq 0$.

Proof: Let $\mathbf{R}$ be the reduced row echelon form of A.

- we will show that $\det(\mathbf{A})$ and $\det(\mathbf{R})$ are both zero or both nonzero.
- Let $\mathbf{E}_1$,$\mathbf{E}_2$, …, $\mathbf{E}_r$ be the elementary matrices that correspond to the elementary row operations that produce $\mathbf{R}$ from $\mathbf{A}$. I.e.
$$\mathbf{R} = \mathbf{E}_r \ldots \mathbf{E}_2\mathbf{E}_1\mathbf{A}.$$
- From (1), $\det(\mathbf{R}) = \det(\mathbf{E}_r) \cdots \det(\mathbf{E}_2)\det(\mathbf{E}_1)\det(\mathbf{A})$
- Note that the determinant of an elementary matrix is **nonzero**.

  Hence, $\det(\mathbf{A})$ and $\det(\mathbf{R})$ are either both zero or both nonzero.

We now prove if and only if statement.

($\Rightarrow$) If we assume first that A is invertible, then $\mathbf{R} = \mathbf{I}$ and hence that $\det(\mathbf{R}) = 1 \neq 0$. This, implies that $\det(\mathbf{A}) \neq 0$, which is what we wanted to show.

(⇐) Assume that $\det(\mathbf{A}) \neq 0$.

- Then $\det(\mathbf{R}) \neq 0$, which means that $\mathbf{R}$ cannot have a row of zeros, which implies that $\mathbf{R} = \mathbf{I}$ (from Chapter 1)
- Finally, $\mathbf{R} = \mathbf{I}$ implies $\mathbf{A}$ is invertible (from Chapter 1).

Theorem 3.5. If $\mathbf{A}$ and $\mathbf{B}$ are square matrices of the same size, then

$$\det(\mathbf{AB}) = \det(\mathbf{A})\det(\mathbf{B})$$

Example 3.1. Verifying $\det(\mathbf{AB}) = \det(\mathbf{A})\det(\mathbf{B})$.

Consider the matrices

$$A = \begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} -1 & 3 \\ 5 & 8 \end{bmatrix}, \quad AB = \begin{bmatrix} 2 & 17 \\ 3 & 14 \end{bmatrix}$$

We leave it for you to verify that

$$\det(A) = 1, \quad \det(B) = -23, \quad \text{and} \quad \det(AB) = -23$$

Theorem 3.6. If $\mathbf{A}$ is invertible, then

$$\det(\mathbf{A}^{-1}) = \frac{1}{\det(\mathbf{A})}$$

Definition 3.1. If $\mathbf{A}$ is any $n \times n$ matrix and $\mathbf{C}_{ij}$ is the cofactor of a_{ij} , then the matrix

$$\begin{bmatrix} C_{11} & C_{12} & \cdots & C_{1n} \\ C_{21} & C_{22} & \cdots & C_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ C_{n1} & C_{n2} & \cdots & C_{nn} \end{bmatrix}$$

is called the **matrix of cofactors** from $\mathbf{A}$. The transpose of this matrix is called the **adjoint** of $\mathbf{A}$ and is denoted by $\text{adj}(\mathbf{A})$.

Theorem 3.7 (Finding Inverse Matrix by Using Its Adjoint)**.** If $\mathbf{A}$ is an invertible matrix, then

$$\mathbf{A}^{-1} = \frac{1}{\det(\mathbf{A})}\text{adj}(\mathbf{A}).$$

Proof: (Sketch) We will show that $\mathbf{A} textrmadj(\mathbf{A}) = \det(\mathbf{A})\mathbf{I}$

Consider $\mathbf{A}\text{adj}(\mathbf{A})$

$$A\,\text{adj}(A) = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{i1} & a_{i2} & \dots & a_{in} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix} \begin{bmatrix} C_{11} & C_{21} & \dots & C_{j1} & \dots & C_{n1} \\ C_{12} & C_{22} & \dots & C_{j2} & \dots & C_{n2} \\ \vdots & \vdots & & \vdots & & \vdots \\ C_{1n} & C_{2n} & \dots & C_{jn} & \dots & C_{nn} \end{bmatrix}$$

and entry is

$$[\mathbf{A}\text{adj}(\mathbf{A})]_{ij} = a_{i1}C_{j1} + a_{i2}C_{j2} + \cdots + a_{in}C_{jn}.$$

- For $i = j$, this is the cofactor expansion of $\det(\mathbf{A})$.
- For $i \neq j$, this is zero.
So $\mathbf{A}\text{adj}(\mathbf{A}) = \det(\mathbf{A})\mathbf{I}$

- Since A is invertible, $\det(\mathbf{A}) \neq 0$, then $\mathbf{A}^{-1} = \frac{1}{\det(\mathbf{A})}\text{adj}(\mathbf{A})$.

Example 3.2. (Adjoint of a matrix **A** and the inverse of **A**).
Let

$$A = \begin{bmatrix} 3 & 2 & -1 \\ 1 & 6 & 3 \\ 2 & -4 & 0 \end{bmatrix}$$

As noted in Example 5, the cofactors of A are

$$\begin{array}{lll} C_{11} = 12 & C_{12} = 6 & C_{13} = -16 \\ C_{21} = 4 & C_{22} = 2 & C_{23} = 16 \\ C_{31} = 12 & C_{32} = -10 & C_{33} = 16 \end{array}$$

so, for example, the cofactor expansion of $\det(A)$ along the first row is

$$\det(A) = 3C_{11} + 2C_{12} + (-1)C_{13} = 36 + 12 + 16 = 64$$

and along the first column is

$$\det(A) = 3C_{11} + C_{21} + 2C_{31} = 36 + 4 + 24 = 64$$

so the matrix of cofactors is

$$\begin{bmatrix} 12 & 6 & -16 \\ 4 & 2 & 16 \\ 12 & -10 & 16 \end{bmatrix}$$

and the adjoint of A is

$$\operatorname{adj}(A) = \begin{bmatrix} 12 & 4 & 12 \\ 6 & 2 & -10 \\ -16 & 16 & 16 \end{bmatrix}$$

$$A^{-1} = \frac{1}{\det(A)}\operatorname{adj}(A) = \frac{1}{64}\begin{bmatrix} 12 & 4 & 12 \\ 6 & 2 & -10 \\ -16 & 16 & 16 \end{bmatrix} = \begin{bmatrix} \frac{12}{64} & \frac{4}{64} & \frac{12}{64} \\ \frac{6}{64} & \frac{2}{64} & -\frac{10}{64} \\ -\frac{16}{64} & \frac{16}{64} & \frac{16}{64} \end{bmatrix}$$

Note:

Suppose, however, we multiply the entries in the first row by the corresponding cofactors from the *second row* and add the resulting products. The result is

$$3C_{21} + 2C_{22} + (-1)C_{23} = 12 + 4 - 16 = 0$$

Or suppose we multiply the entries in the first column by the corresponding cofactors from the *second column* and add the resulting products. The result is again zero since

$$3C_{12} + 1C_{22} + 2C_{32} = 18 + 2 - 20 = 0$$

Theorem 3.8 (Cramer's Rule). If $\mathbf{Ax} = \mathbf{b}$ is a system of n linear equations in n unknowns such that $\det(\mathbf{A}) \neq 0$, then the system has a unique solution. This solution is

$$x_1 = \frac{\det(\mathbf{A}_1)}{\det(\mathbf{A})}, \qquad x_2 = \frac{\det(\mathbf{A}_2)}{\det(\mathbf{A})} \cdots \qquad x_n = \frac{\det(\mathbf{A}_n)}{\det(\mathbf{A})},$$

where A_j is the matrix obtained by replacing the entries in the j th column of $\mathbf{A}$ by the entries in the matrix $\mathbf{b} = [b_1, b_2, \ldots, b_n]^T$

<u>Proof:</u> Since $\det(\mathbf{A}) \neq 0$, then $\mathbf{A}$ is invertible and the solution of $\mathbf{Ax} = \mathbf{b}$ is

$$\mathbf{x} = A^{-1}\mathbf{b} = \frac{1}{\det(A)}\operatorname{adj}(A)\mathbf{b} = \frac{1}{\det(A)} \begin{bmatrix} C_{11} & C_{21} & \cdots & C_{n1} \\ C_{12} & C_{22} & \cdots & C_{n2} \\ \vdots & \vdots & & \vdots \\ C_{1n} & C_{2n} & \cdots & C_{nn} \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

$$\mathbf{x} = \frac{1}{\det(A)} \begin{bmatrix} b_1C_{11} + b_2C_{21} + \cdots + b_nC_{n1} \\ b_1C_{12} + b_2C_{22} + \cdots + b_nC_{n2} \\ \vdots \qquad \vdots \qquad \vdots \\ b_1C_{1n} + b_2C_{2n} + \cdots + b_nC_{nn} \end{bmatrix}$$

The entry in the jth row of $\mathbf{x}$ is therefore

$$x_j = \frac{b_1C_{1j} + b_2C_{2j} + \cdots + b_nC_{nj}}{\det(A)}$$

Now let

$$A_j = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1j-1} & b_1 & a_{1j+1} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2j-1} & b_2 & a_{2j+1} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nj-1} & b_n & a_{nj+1} & \cdots & a_{nn} \end{bmatrix}$$

Since A_j differs from A only in the jth column, it follows that the cofactors of entries $b_1, b_2, \ldots, b_n$ in A_j are the same as the cofactors of the corresponding entries in the jth column of A. The cofactor expansion of $\det(A_j)$ along the jth column is therefore

$$\det(A_j) = b_1C_{1j} + b_2C_{2j} + \cdots + b_nC_{nj}$$

That is,

$$x_j = \frac{\det(A_j)}{\det(A)}$$

.

Cramer's Rule

- For $n = 2$, $\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$ Cramer's Rule:

$$x_1 = \frac{\begin{vmatrix} \boxed{b_1} & a_{12} \\ \boxed{b_2} & a_{22} \end{vmatrix}}{\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}} = \frac{b_1 a_{22} - b_2 a_{12}}{a_{11}a_{22} - a_{21}a_{12}}, \quad x_2 = \frac{\begin{vmatrix} a_{11} & \boxed{b_1} \\ a_{21} & \boxed{b_2} \end{vmatrix}}{\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}} = \frac{a_{11}b_2 - a_{21}b_1}{a_{11}a_{22} - a_{21}a_{12}}$$

- For $n = 3$, $\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$

$$x_1 = \frac{\begin{vmatrix} \boxed{b_1} & a_{12} & a_{13} \\ \boxed{b_2} & a_{22} & a_{23} \\ \boxed{b_2} & a_{32} & a_{33} \end{vmatrix}}{\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}}, \quad x_2 = \frac{\begin{vmatrix} a_{11} & \boxed{b_1} & a_{13} \\ a_{21} & \boxed{b_2} & a_{23} \\ a_{31} & \boxed{b_2} & a_{33} \end{vmatrix}}{\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}}, \quad x_3 = \frac{\begin{vmatrix} a_{11} & a_{12} & \boxed{b_1} \\ a_{21} & a_{22} & \boxed{b_2} \\ a_{31} & a_{32} & \boxed{b_2} \end{vmatrix}}{\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}}$$

Example 3.3. Use Cramers rule to solve the following linear system of equations.

$$\begin{aligned} x_1 + \quad + 2x_3 &= 6 \\ -3x_1 + 4x_2 + 6x_3 &= 30 \\ -x_1 - 2x_2 + 3x_3 &= 8 \end{aligned}$$

<u>**Solution:**</u>

$$A = \begin{bmatrix} 1 & 0 & 2 \\ -3 & 4 & 6 \\ -1 & -2 & 3 \end{bmatrix}, \quad A_1 = \begin{bmatrix} 6 & 0 & 2 \\ 30 & 4 & 6 \\ 8 & -2 & 3 \end{bmatrix},$$

$$A_2 = \begin{bmatrix} 1 & 6 & 2 \\ -3 & 30 & 6 \\ -1 & 8 & 3 \end{bmatrix}, \quad A_3 = \begin{bmatrix} 1 & 0 & 6 \\ -3 & 4 & 30 \\ -1 & -2 & 8 \end{bmatrix}$$

$$x_1 = \frac{\det(A_1)}{\det(A)} = \frac{-40}{44} = \frac{-10}{11}, \quad x_2 = \frac{\det(A_2)}{\det(A)} = \frac{72}{44} = \frac{18}{11}, \quad x_3 = \frac{\det(A_3)}{\det(A)} = \frac{152}{44} = \frac{38}{11}$$

Theorem 3.9. Equivalent Statements

If $\mathbf{A}$ is an $n \times n$ matrix, then the following statements are equivalent.

(a) $\mathbf{A}$ is invertible.

(b) $\mathbf{Ax} = 0$ has only the trivial solution.

(c) The reduced row echelon form of $\mathbf{A}$ is In.

(d) $\mathbf{A}$ can be expressed as a product of elementary matrices.

(e) $\mathbf{Ax} = \mathbf{b}$ is consistent for every $n \times 1$ matrix $\mathbf{b}$.

(f) $\mathbf{Ax} = \mathbf{b}$ has exactly one solution for every $n \times 1$ matrix b.

(g) $\det(\mathbf{A}) \neq 0$.