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Question 0:

Consider the function f defined for all (x,y) such that

$$f\,(x, y; a) = \frac{1}{2}x^2 - x + ay(x-1) - \frac{1}{3}y^3 + a^2y^2,$$

where a is a constant.

a. Prove that $(x^*, y^*) = (1 - a^3, a^2)$ is a stationary point of $f\,(x, y; a)$.
b. State the condition under which the above stationary point is a global maximum.
c. Given that $G(a) = f\,(x^*, y^*; a)$, show the derivative of G with respect to a.

a) $\frac{df}{dx} = x - 1 + ay = 0 \longrightarrow x = 1 - ay$

$\frac{df}{dy} = ax - a - y^2 + 2a^2y = 0 \longrightarrow y^2 - 2a^2y = ax - a$

$y^2 - 2a^2y = a(1 - ay) - a$

$y^2 - 2a^2y + a^2y = 0$

$y(y - 2a^2 + a^2) = 0$

$y^* = 0,\ a^2$

$x^* = 0,\ 1 - a^3$

$\therefore$ stationary point of $f(x, y; a)$ is $(x^*, y^*) = (1 - a^3, a^2)$

b) $H = \begin{bmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{bmatrix} = \begin{bmatrix} 1 & a \\ a & -2y + 2a^2 \end{bmatrix}$

global maximum means that always $|H_1| < 0$, $|H_2| > 0$, d^2f is always < 0

$|H_1| = 1$

$|H_2| = (-2y + 2a^2) - a^2 = -2y + a^2$

$= -2(a^2) + a^2$

$= -a^2$

$\therefore$ In this case $|H_1| > 0$, $|H_2| < 0$, so $f(x, y; a)$ is inconclusive.

c) $G(a) = \frac{1}{2}(1 - a^3)^2 - (1 - a^3) + a(a^2)(\cancel{1} - a^3 - \cancel{1}) - \frac{1}{3}(a^2)^3 + a^2(a^2)^2$

$= \frac{1}{2}(1 - 2a^3 + a^6) - 1 + a^3 - \cancel{a^6} - \frac{1}{3}a^6 + \cancel{a^6}$

$= \frac{1}{2} - \cancel{a^3} + \frac{a^6}{2} - 1 + \cancel{a^3} - \frac{a^6}{3} = -\frac{1}{2} + \frac{a^6}{2} - \frac{a^6}{3}$

$\frac{dG(a)}{da} = 3a^5 - 2a^5 = a^5$ //

d. Calculate $\frac{\partial f\,(x,y;a)}{\partial a}$ and evaluate its value where $(x^*, y^*) = (1 - a^3, a^2)$. Compare your answer with the answer obtained in b. Are they the same?

e. Determine the domain of (x,y) in the xy-plan where f is convex.

d. Calculate $\frac{\partial f(x,y;a)}{\partial a}$

$$\frac{\partial f(x,y;a)}{\partial a} = y(x-1) + 2ay^2$$

$$\left.\frac{\partial f(x,y;a)}{\partial a}\right|_{x^*=1-a^3,\; y^*=a^2} = a^2(1-a^3-1) + 2a(a^2)^2$$

$$= -a^5 + 2a^5$$

$$= a^5$$

This answer is different from the answer obtained in b.

e. For the function f to be convex, Hessian matrix must be positive definite.

From (b), $|H_1| = 1 > 0$

$$|H_2| = -2y + a^2 > 0$$

$$-2y > -a^2$$

$$y < \frac{a^2}{2}$$

Domain $y = \left(-\infty, \frac{a^2}{2}\right)$

$x = 1 - ay$

For $y \to -\infty$: $x \to +\infty$

For $y = \frac{a^2}{2}$: $x = 1 - a\left(\frac{a^2}{2}\right) = 1 - \frac{a^3}{2}$

Domain $x = \left(1 - \frac{a^3}{2}, +\infty\right)$

Question 1

Consider a linear demand equation faced by a monopolist.

$$Q = 2000 + 4\sqrt{A} - 20P,$$

where A is the dollar amount of the expenditure on advertisement, P is the unit price, and Q is the amount of quantity demanded by consumers. The demand equation is an extended version of the traditional one. The proposed function captures an idea that advertising can boost the total demand in the market as potential customers get more information about the product.

Suppose that the monopolist cost function is given by $C(Q, A) = c(Q) + A$ where $c(Q) = 2Q + 1000$

Consider the following problem.

a) Construct the profit function.
b) Determine the optimal pricing (P*) and optimal advertisement (A*) that maximize the profit of the monopolist.
c) Confirm the result with the second-order differential test, i.e. hessian-matrix method.
d) What is the maximum level of profit?

a) $\pi = P \times Q - C(Q, A)$

$$= P(2{,}000 + 4\sqrt{A} - 20P) - \left[2Q + 1{,}000 + A\right]$$

$$\pi = 2{,}000P + 4\sqrt{A}P - 20P^2 - 2(2{,}000 + 4\sqrt{A} - 20P) - 1{,}000 - A$$

$$= 2{,}000P + 4\sqrt{A}P - 20P^2 - 4{,}000 - 8\sqrt{A} + 40P - 1{,}000 - A$$

$$= -5{,}000 + 4\sqrt{A}P - 8\sqrt{A} - A + 2{,}040P - 20P^2$$

b) F.O.C

$$\frac{\partial \pi}{\partial P} = \frac{\partial(-5{,}000 + 4\sqrt{A}P - 8\sqrt{A} - A + 2{,}040P - 20P^2)}{\partial P}$$

$$= 4A^{\frac{1}{2}} + 2040 - 40P = 0 \longrightarrow A^{\frac{1}{2}} = \frac{40P - 2040}{4}$$

$$A^{\frac{1}{2}} = 10P - 510$$

$$\frac{1}{A^{\frac{1}{2}}} = \frac{1}{10P - 510} \quad ①$$

$$\frac{\partial \pi}{\partial A} = \frac{\partial(-5{,}000 + 4\sqrt{A}P - 8\sqrt{A} - A + 2{,}040P - 20P^2)}{\partial A}$$

$$= 2A^{-\frac{1}{2}}P - 4A^{-\frac{1}{2}} - 1 = 0 \longrightarrow (A^{-\frac{1}{2}})(2P - 4) = 1$$

$$\frac{1}{A^{\frac{1}{2}}} = \frac{1}{2P - 4} \quad - ②$$

① = ②; $\dfrac{1}{10P - 510} = \dfrac{1}{2P - 4}$

$$2P - 4 = 10P - 510$$

$$2P - 10P = -510 + 4$$

$$8P = 506$$

$$P = 63.25$$

Sub $P = 63.25$ into the equation

$$A^{\frac{1}{2}} = 10(63.25) - 510$$

$$A^{\frac{1}{2}} = 122.5$$

$$A = 15{,}006.25$$

(c) 2^{nd} order condition check $d^2 \pi$

$$H = \begin{bmatrix} \pi_{PP} & \pi_{PA} \\ \pi_{AP} & \pi_{AA} \end{bmatrix}$$

$$\pi_{PP} = \frac{\partial (4A^{\frac{1}{2}} + 2040 - 40P)}{\partial P}$$

$$= -40$$

$$\pi_{PA} = \frac{\partial (4A^{\frac{1}{2}} + 2040 - 40P)}{\partial A}$$

$$= 2A^{-\frac{1}{2}}$$

$$\pi_{AP} = \frac{\partial (2A^{-\frac{1}{2}}P - 4A^{-\frac{1}{2}} - 1)}{\partial P}$$

$$= 2A^{-\frac{1}{2}}$$

$$\pi_{AA} = \frac{\partial (2A^{-\frac{1}{2}}P - 4A^{-\frac{1}{2}} - 1)}{\partial A}$$

$$= -A^{-\frac{3}{2}}P + 2A^{-\frac{3}{2}}$$

$$H = \begin{bmatrix} -40 & 2A^{-\frac{1}{2}} \\ 2A^{-\frac{1}{2}} & -A^{-\frac{3}{2}}P + 2A^{-\frac{3}{2}} \end{bmatrix}$$

$$|H_1| = -40$$

$$|H_2| = (-40)(-A^{-\frac{3}{2}}P + 2A^{-\frac{3}{2}}) - (2A^{-\frac{1}{2}})(2A^{-\frac{1}{2}})$$

$$= 40A^{-\frac{3}{2}}P - 80A^{-\frac{3}{2}} - 4A^{-1}$$

As $|H_1| < 0$ and $|H_2| > 0$ (with the alternative sign)
H is negative definite $\rightarrow d^2\pi < 0$ or π is concave at the solution obtained from F.O.C (63.25, 15,006.25)
not only local but also global sol^n B/c π is globally concave f^n

d) $\pi(P, A) = -5{,}000 + 4\sqrt{A}P - 8\sqrt{A} - A + 2{,}040P - 20P^2$

$$= -5{,}000 + (4 \times \sqrt{15{,}006.25} \times 63.25) - (8\sqrt{15{,}006.25}) - 15{,}006.25 + (2{,}040 \times 63.25) - 20(63.25)^2$$

$$= 59{,}025$$

the maximum level of profit is 59,025

Question 3:

Consider a simple two-market model where demand and supply for each market is given below. (Notationally, let's name the two markets as A and B, respectively.)

Market A:

Demand: $p_A^d = 10 - 2Q_A^d \Rightarrow Q_A^d = \frac{10}{2} - \frac{p_A^d}{2}$

Supply: $p_A^s = 1 + Q_A$

Market B:

Demand: $p_B = 20 - Q_B$

Supply: $p_B = 2 + 2Q_B$

a. Derive the market equilibrium
b. Suppose the government imposes unit tax on consumers in both markets at the rate of t_A and t_B. Solve for the after-tax equilibrium as the function of t_A and t_B.
c. How much revenue can the government collect from the taxation?

a. Mkt A

$D = S$

$10 - 2Q_A = 1 + Q_A$

$9 = 3Q_A \Rightarrow Q_A = 3$

$P_A = 1 + Q_A$

$= 1 + (3)$

$P_A = 4$

$\therefore$ Market A equilibrium is $P_A^* = ฿4$, $Q_A^* = 3$ units.

Mkt B

$D = S$

$20 - Q_B = 2 + 2Q_B$

$18 = 3Q_B$

$Q_B^* = 6$

$P_B = 20 - Q_B$

$= 20 - (6)$

$P_B^* = 14$

$\therefore$ Market B equilibrium is $P_B^* = ฿14$, $Q_B^* = 6$ units

b. Mkt A

$Q_A^d = Q_A^s$

$5 - \frac{1}{2}p_A^d = p_A^s - 1$

$5 - \frac{1}{2}(p_A^s + t_A) = p_A^s - 1$

$6 - \frac{1}{2}t_A = \frac{3}{2}p_A^s$

$p_A^s = 4 - \frac{1}{3}t_A$

$p_A^d = p_A^s + t_A = 4 - \frac{1}{3}t_A + t_A = 4 + \frac{2}{3}t_A$

$Q_A^* = p_A^s - 1 = 3 - \frac{1}{3}t_A$

Mkt B

$p_B^s = 2 + 2Q_B^s \Rightarrow Q_B^s = \frac{1}{2}p_B^s - 1$

$p_B^d = 20 - Q_B^d \Rightarrow Q_B^d = 20 - p_B^d$

$Q_B^d = Q_B^s$

$\frac{1}{2}p_B^s - 1 = 20 - p_B^d$

$\frac{1}{2}p_B^s - 1 = 20 - (p_B^s + t_B)$

$\frac{3}{2}p_B^s = 21 - t_B$

$p_B^s = 14 - \frac{2}{3}t_B$

$p_B^d = p_B^s + t_B = 14 + \frac{1}{3}t_B$

$Q_B^* = 20 - p_B^d = 6 - \frac{1}{3}t_B$

c. Mkt A

Tax Rev $= t_A \cdot Q_A^*$

$= t_A(3 - \frac{1}{3}t_A) = 3t_A - \frac{1}{3}t_A^2$ #

Mkt B

Tax Rev. $= t_B \cdot Q_B^*$

$= t_B(6 - \frac{1}{3}t_B) = 6t_B - \frac{1}{3}t_B^2$ #

$\therefore$ Gov. can collect tax

from mkt A = ฿ $3t_A - \frac{1}{3}t_A^2$

from mkt B = ฿ $6t_B - \frac{1}{3}t_B^2$

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Supply: $p_A = 1 + Q_A$

Market B:

Demand: $p_B = 20 - Q_B$

Supply: $p_B = 2 + 2Q_B$

a. Derive the market equilibrium
b. Suppose the government imposes unit tax on consumers in both markets at the rate of t_A and t_B. Solve for the after-tax equilibrium as the function of t_A and t_B.
c. How much revenue can the government collect from the taxation?
d. Determine the level of t_A and t_B that maximizes government's revenue.
e. Use the second-order conditions test and show that the answer obtained in "d" is a global solution.

d. Determine the level of t_A and t_B that maximizes government revenue

Tax revenue : T = tax from market A + tax from market B

$$T = 3t_A - \frac{1}{3}t_A^2 + 6t_B - \frac{1}{3}t_B^2$$

F.O.C :

$$[t_A] : \frac{\partial T}{\partial t_A} = 3 - \frac{2}{3}t_A = 0$$

$$\Rightarrow -\frac{2}{3}t_A = -3$$

$$t_A = \frac{3}{\frac{2}{3}} = \frac{9}{2} \text{ units} = 4.5 \text{ units}$$

$$[t_B] : \frac{\partial T}{\partial t_B} = 6 - \frac{2}{3}t_B = 0$$

$$\Rightarrow -\frac{2}{3}t_B = -6$$

$$t_B = \frac{6}{\frac{2}{3}} = \frac{18}{2} = 9 \text{ units}$$

e.

S.O.C:

$$H = \begin{bmatrix} T_{AA} & T_{AB} \\ T_{BA} & T_{BB} \end{bmatrix} = \begin{bmatrix} -\frac{2}{3} & 0 \\ 0 & -\frac{2}{3} \end{bmatrix}$$

$$|H_1| = \left|-\frac{2}{3}\right| = -\frac{2}{3} < 0 \quad , \forall t_A , t_B$$

$$|H_2| = \begin{vmatrix} -\frac{2}{3} & 0 \\ 0 & -\frac{2}{3} \end{vmatrix} = \left(-\frac{2}{3}\right)\left(-\frac{2}{3}\right) - 0$$

$$= \frac{4}{9} > 0 \quad , \forall t_A , t_B$$

Since $|H_i|$ has alternative sign with negative first, H is always negative definite, $\forall t_A , t_B$.

$d^2T < 0 \quad , \forall t_A , t_B$

=> T is globally concave function.

=> The solution obtained in "d" is not only local solution but it's global solution.