Exercise: Partial Fraction Decomposition

1. Write out the appropriate form of the partial fraction decomposition of the given expressions Do not evaluate the coefficients.

   (a)
$$\frac{x+1}{2x^2-11x+15}$$

   Answer:
$$\frac{x+1}{2x^2-11x+15} = \frac{x+1}{(x-3)(2x-5)} = \frac{A}{x-3} + \frac{B}{2x-5},$$
   where A, B are constants.

   (b)
$$\frac{1}{x^4(x-1)(x^2+4)^3(x^2+x+1)^2}$$

   Answer: Note that, x^2+4 and x^2+x+1 are irreducible polynomial (this can be checked by using $b^2-4ac<0$ for ax^2+bx+c). So the partial fraction decomposition of $\frac{1}{x^4(x-1)(x^2+4)^3(x^2+x+1)^2}$ is in the form
$$= \frac{A_1}{x}+\frac{A_2}{x^2}+\frac{A_3}{x^3}+\frac{A_4}{x^4}+\frac{B}{x-1}+\frac{C_1x+D_1}{x^2+4}+\frac{C_2x+D_2}{(x^2+4)^2}+\frac{C_3x+D_3}{(x^2+4)^3}+\frac{E_1x+F_1}{x^2+x+1}+\frac{E_2x+F_2}{(x^2+x+1)^2},$$
   where $A_1, A_2, A_3, A_4, B, C_1, C_2, C_3, D_1, D_2, D_3, E_1, E_2, F_1, F_2$ are constants.

2. Determine the partial fraction decomposition of
$$\frac{x^4+7x^3+8x^2-3}{x^3+6x^2}.$$

   Answer: Note that the given expression is not in the form of proper fraction. Using the long division, we have
$$\frac{x^4+7x^3+8x^2-3}{x^3+6x^2} = x+1+\frac{2x^2-3}{x^2(x+6)}$$
   So we will apply partial fraction decomposition on $\frac{2x^2-3}{x^2(x+6)}$ which will be in the form:
$$\frac{2x^2-3}{x^2(x+6)} = \frac{A}{x}+\frac{B}{x^2}+\frac{C}{x+6}.$$
   We can solve for A, B, C as follows.
$$\begin{aligned}
\frac{2x^2-3}{x^2(x+6)} &= \frac{A}{x}+\frac{B}{x^2}+\frac{C}{x+6} = \frac{Ax(x+6)+B(x+6)+Cx^2}{x^2(x+6)} \\
2x^2-3 &= Ax(x+6)+B(x+6)+Cx^2 \\
&= \underbrace{[A+C]}_{=2}x^2+\underbrace{[6A+B]}_{=0}x+\underbrace{6B}_{=-3},
\end{aligned}$$

which gives 3 equations

$$A + C = 2 \qquad 6A + B = 0 \qquad 6B = -3$$

and so $A = 1/12, B = -1/2, C = 23/12$. So, the partial fraction decomposition is

$$\frac{x^4 + 7x^3 + 8x^2 - 3}{x^3 + 6x^2} = x + 1 + \frac{1}{12x} - \frac{1}{2x^2} + \left(\frac{23}{12}\right)\frac{1}{x+6}.$$

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Optional problems

Determine the partial fraction decomposition of the following rational functions.

1. $\frac{6x-1}{2x^4-x^3}$

Answer: First, we factor the denominator:

$$2x^4 - x^3 = x^3(2x - 1).$$

Since x is a repeated liner factor with the distinct linear factor $2x - 1$ in the denominator, the partial fraction decomposition is in the form

$$\frac{6x - 1}{x^3(2x - 1)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x^3} + \frac{D}{2x - 1}.$$

To find the constants A, B, C, D, consider

$$\frac{6x - 1}{x^3(2x - 1)} = \frac{Ax^2(2x - 1) + Bx(2x - 1) + C(2x - 1) + Dx^3}{x^3(2x - 1)}$$

and equate the numerators:

$$\begin{aligned} 6x - 1 &= Ax^2(2x - 1) + Bx(2x - 1) + C(2x - 1) + Dx^3 \\ &= (2A + D)x^3 + (-A + 2B)x^2 + (-B + 2C)x - C. \end{aligned}$$

First,

- set$x = 0$, we have $0 - 1 = C(0 - 1)$ which gives $C = 1$,
- set $x = 1/2$, we have $6 \cdot \frac{1}{2} - 1 = D(1/2)^3$ which gives $D = 16$.

Then we compare the coefficients of the terms with the same power of x:

$$2A + D = 0, \quad -A + 2B = 0, \quad -B + 2C = 6, \quad -C = -1.$$

Substituting $C = 1$ and $D = 16$, we have $A = -D/2 = -8$, $B = A/2 = -4$. That is, the partial fraction decomposition is

$$\frac{6x - 1}{x^3(2x - 1)} = \frac{-8}{x} + \frac{-4}{x^2} + \frac{1}{x^3} + \frac{16}{2x - 1}.$$

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2. $\frac{x+3}{x^4+9x^2}$

Answer: First, we factor the denominator:

$$x^4 + 9x^2 = x^2(x^2 + 9).$$

Since x is a repeated liner factor with the distinct linear factor $x^2 + 9$ in the denominator, the partial fraction decomposition is in the form

$$\frac{x+3}{x^2(x^2+9)} = \frac{A}{x} + \frac{B}{x^2} + \frac{Cx+D}{x^2+9}.$$

To find the constants A, B, C, D, consider

$$\frac{x+3}{x^2(x^2+9)} = \frac{Ax(x^2+9) + B(x^2+9) + (Cx+D)x^2}{x^2(x^2+9)}$$

and equate the numerators:

$$\begin{aligned} x+3 &= Ax(x^2+9) + B(x^2+9) + (Cx+D)x^2 \\ &= (A+C)x^3 + (B+D)x^2 + (9A)x + 9B. \end{aligned}$$

By comparing the coefficients of the terms with the same power of x:

$$A + C = 0, \quad B + D = 0, \quad 9A = 1, \quad 9B = 3.$$

That is,

$$A = \frac{1}{9}, \quad B = \frac{1}{3}, \quad C = -\frac{1}{9}, \quad D = -\frac{1}{3}.$$

Hence, the partial fraction decomposition is

$$\frac{x+3}{x^2(x^2+9)} = \frac{1}{9x} + \frac{1}{3x^2} - \frac{1}{9}\frac{x+3}{(x^2+9)}.$$

■

3. $\frac{x^3-2x}{x^2+3x+2}$

Answer: Since the power of the numerator is not less than the power of the denominator, we first have to divide the numerator by the denominator (e.g. long division) to obtain

$$\frac{x^3-2x}{x^2+3x+2} = x - 3 + \frac{5x+6}{x^2+3x+2},$$

so that we can apply the partial fraction decomposition for $\frac{5x+6}{x^2+3x+2}$. Since $x^2 + 3x + 2 = (x+1)(x+2)$, the decomposition is in the form

$$\frac{5x+6}{(x+1)(x+2)} = \frac{A}{x+1} + \frac{B}{x+2}$$

$$\frac{5x+6}{(x+1)(x+2)} = \frac{A(x+2) + B(x+1)}{(x+1)(x+2)}$$

$$5x+6 = A(x+2)+B(x+1)$$

By setting $x=-1$, $5(-1)+6 = A(-1+2)$, which gives $A=1$ By setting $x=-2$, $5(-2)+6 = B(-2+1)$, which gives $B=4$. That is,

$$\frac{x^3-2x}{x^2+3x+2} = x-3+\frac{1}{x+1}+\frac{4}{x+2}.$$

■

4. $\frac{x^2}{(x^2+4)}$

Answer:

Since the power of the numerator is not less than the power of the denominator, we first have to divide the numerator by the denominator (e.g. long division) to obtain

$$\frac{x^2}{(x^2+4)} = 1-\frac{4}{x^2+4}$$

Since the divisor is an irreducible second-order polynomial and the remainder is just a constant, the remainder over the divisor, $-\frac{4}{x^2+4}$, is already in the form of partial fraction decomposition and above is the final answer. ■

5. $\frac{4x}{(x^2+1)(x^2+2x+3)}$

Answer: Since the denominator is already factored into the product of irreducible polynomials, the partial fraction decomposition is in the form

$$\frac{4x}{(x^2+1)(x^2+2x+3)} = \frac{Ax+B}{x^2+1}+\frac{Cx+D}{x^2+2x+3}$$

from which we find

$$\begin{aligned} 4x &= (Ax+B)(x^2+2x+3)+(Cx+D)(x^2+1) \\ &= (A+C)x^2+(2A+B+D)x^2+(3A+2B+C)x+(3B+D). \end{aligned}$$

By comparing the coefficients of the powers of x:

$$\begin{aligned} A+C &= 0 \\ 2A+B+D &= 0 \\ 3A+2B+C &= 4 \\ 3B+D &= 0 \end{aligned}$$

which give $A=1$, $B=1$, $C=-1$, and $D=-3$. Therefore the partial fraction decomposition is

$$\frac{4x}{(x^2+1)(x^2+2x+3)} = \frac{x+1}{x^2+1}-\frac{x+3}{x^2+2x+3}.$$

■