

Tutorial 3

Matrix Algebra

1. Find all solution to the following system of equations

(a)

$$3x + 4y + z = 1$$

$$2x + 3y = 0$$

$$4x + 3y - z = -2$$

(Ans: $x=-3/7, y=2/7, z=8/7$)

$$x + 2y - z = 2$$

(b) $2x + 5y + 2z = -1$

$$7x + 17y + 5z = -1$$

Ans:

$$\begin{aligned} x &= 12 + 9z \\ y &= -5 - 4z \end{aligned} \quad (\text{parametric form})$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 12 \\ -5 \\ 0 \end{bmatrix} + C \begin{bmatrix} 9 \\ -4 \\ 1 \end{bmatrix}; C \in \mathbb{R} \quad (\text{vectors form})$$

$$x + 10z = 5$$

c) $3x + y - 4z = -1$

$$4x + y + 6z = 1$$

(ans: No solution)

2. Compute the rank of $A = \begin{bmatrix} 1 & 1 & -1 & 4 \\ 2 & 1 & 3 & 0 \\ 0 & 1 & -5 & 8 \end{bmatrix}$

(ans: 2)

3. Find a condition on the number a, b, c such that the following system of equations is consistent. When that condition is satisfied, find all solutions (in term of a, b, c)

$$x + 3y + z = a$$

$$-x - 2y + z = b$$

$$3x + 7y - z = c$$

Ans: Consistent if $c=a-2b$

$$x = 5z - (2a + 3b)$$

$$y = (a + b) - 2z \quad (\text{Parametric form})$$

Z = free variables

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -(2a + 3b) \\ a + b \\ 0 \end{bmatrix} + C \begin{bmatrix} 5 \\ -2 \\ 1 \end{bmatrix} \quad C \in \mathbb{R} \quad (\text{vectors form})$$

4. Suppose a square matrix A satisfies $A = 2A^T$. Show that necessarily $A=0$.

Ans:

$$A = 2(2A^T)^T = 2(2(A^T)^T) = 4A$$

$$3A = 0$$

$$A = 0$$

5. In each case find the matrix A

$$(a) \left(3A^T + 2 \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \right)^T = \begin{bmatrix} 8 & 0 \\ 3 & 1 \end{bmatrix}$$

$$\text{Ans: } A = \begin{bmatrix} 2 & 0 \\ 1 & -1 \end{bmatrix}$$

$$(b) \left(2A^T - 5 \begin{bmatrix} 1 & 0 \\ -1 & 2 \end{bmatrix} \right)^T = 4A - 9 \begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix}$$

$$\text{Ans: } A = \begin{bmatrix} 2 & 7 \\ -9 & -5 \end{bmatrix}$$

6. Find the inverse of the matrix

$$A = \begin{bmatrix} 2 & 7 & 1 \\ 1 & 4 & -1 \\ 1 & 3 & 0 \end{bmatrix}$$

$$\text{Ans: } A^{-1} = \frac{1}{2} \begin{bmatrix} -3 & -3 & 11 \\ 1 & 1 & -3 \\ 1 & -1 & -1 \end{bmatrix}$$

$$7. \text{ Find } A \text{ if } (A^T - 2I)^{-1} = \begin{bmatrix} 2 & 1 \\ -1 & 0 \end{bmatrix}$$

$$\text{Ans: } A = \begin{bmatrix} 2 & 1 \\ -1 & 4 \end{bmatrix}$$

8. Suppose A is 3 by 5, B is 5 by 3, C is 5 by 1, and D is 3 by 1. Which of these matrix operations are allowed, and what are the shape of the results?

$BA, A(B+C), ABD, AC+BD, ABABD$

(Ans: **BA** (a 5 x5 matrix) **ABD** (a 3x1 matrix) and **ABABD** (a 3x1 matrix))

9. Compute AB where $A = \begin{bmatrix} 2 & 0 & -3 \\ -1 & 6 & 5 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 3 \\ -4 & 7 \\ 2 & -5 \end{bmatrix}$. Show how you obtain the result using all three methods you study in the class.

$$\text{Ans: } \begin{bmatrix} -6 & 21 \\ -14 & 14 \end{bmatrix}$$

10. Find all solutions to the following system of linear equations:

$$\begin{aligned} x + y - 3z &= 3 \\ -2x - y &= -4 \\ 4x + 2y + 3z &= 7 \end{aligned}$$

$$\text{Ans: } x=2, y=0, z = -1/3$$

11. Determine the value(s) of h such that the matrix is the augmented matrix of a consistent linear system

$$\text{a) } \left[\begin{array}{cc|c} 2 & 3 & h \\ 4 & 6 & 7 \end{array} \right] \quad (\text{Ans: } h=7/2)$$

$$\text{b) } \left[\begin{array}{cc|c} 1 & -3 & 2 \\ 5 & h & -7 \end{array} \right] \quad (\text{Ans: } h \neq -15)$$

12. For matrix

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 4 & 0 \\ 0 & 2 & 8 & 0 \end{bmatrix}$$

determine the echelon form U , the basic variables, the free variables and the general solution to $\mathbf{Ax}=0$. Then apply, elimination to $\mathbf{Ax}=\mathbf{b}$, with b_1 and b_2 on the right side; find the conditions for $\mathbf{Ax}=\mathbf{b}$ to be consistent and find the general solution. What is the rank of $\mathbf{A}$?

$$\text{Ans: For } \mathbf{Ax}=0 \rightarrow \mathbf{x} = C_1 \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + C_3 \begin{bmatrix} 0 \\ -4 \\ 1 \\ 0 \end{bmatrix} + C_4 \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \text{ where } C_1, C_3, C_4 \in \mathbb{R}$$

For $\mathbf{Ax}=\mathbf{b} \rightarrow$

$$\mathbf{x} = \begin{bmatrix} 0 \\ b_1 \\ 0 \\ 0 \end{bmatrix} + C_1 \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + C_3 \begin{bmatrix} 0 \\ -4 \\ 1 \\ 0 \end{bmatrix} + C_4 \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \text{ where } C_1, C_3, C_4 \in \mathbb{R}$$

13. a) Find all solutions to

$$\mathbf{Ux} = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\text{Ans: } \mathbf{x} = C_1 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + C_2 \begin{bmatrix} 2 \\ 0 \\ -2 \\ 1 \end{bmatrix} \text{ where } C_1, C_2 \in \mathbb{R}$$

b) If the right side is changed from $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ to $\begin{bmatrix} a \\ b \\ 0 \end{bmatrix}$, what are the solutions?

$$\text{Ans: } \mathbf{x} = \begin{bmatrix} a-3b \\ 1 \\ b \\ 0 \end{bmatrix} + C_1 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + C_2 \begin{bmatrix} 2 \\ 0 \\ -2 \\ 1 \end{bmatrix} \text{ where } C_1, C_2 \in \mathbb{R}$$

14. Use row operation to find

$$6x_1 + x_2 + x_3 = 6$$

$$5x_1 + x_2 + 2x_3 = 4$$

$$4x_1 + x_2 - x_3 = -2$$

$$\text{Ans: } \begin{bmatrix} 3 \\ -13 \\ 1 \end{bmatrix}$$

15. Suppose

$$Ax = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \quad \text{has no solution}$$

but

$$Ax = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} \quad \text{has infinitely many solutions.}$$

- a) Find all possible information about r , m and n . (r is the rank and A is of the size $m \times n$)
- b) Find an example of such a matrix A with r , m and n all as small as possible.

Ans:

a) $m=3, 0 < r < m$ and $r < n$

b) $r=1, m=3, n=2; \quad A = \begin{bmatrix} 3 & 0 \\ 2 & 0 \\ 1 & 0 \end{bmatrix}$

16. a) By elimination put A into its upper triangular form U . Which are the pivot columns and free columns?

$$A = \begin{bmatrix} 1 & 3 & 2 & 1 \\ 2 & 8 & 5 & 2 \\ 1 & 5 & 3 & 1 \end{bmatrix}$$

(Ans: pivot columns 1,2 and free columns 3,4)

b) Describe specifically the vectors which are solutions to $Ax=0$

- c) Does $Ax=b$ have a solution for the right side $b = \begin{bmatrix} 3 \\ 8 \\ 5 \end{bmatrix}$? If it does, find one particular solution and then complete solution to this system $Ax=b$.

Ans:

$$x = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + c \begin{bmatrix} -\frac{1}{2} \\ -\frac{1}{2} \\ 1 \\ 0 \end{bmatrix} + d \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

17. a) Show that $A = \begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix}$ has inverse $C = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$

Show that $A = \begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix}$ has inverse $C = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$.

We verify that $AC = I$ and $CA = I$:

$$AC = \begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

and

$$CA = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

b) Show that $A = \begin{bmatrix} 2 & -3 \\ 0 & 0 \end{bmatrix}$ has no inverse.

If $C = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is any matrix, then $AC = \begin{bmatrix} 2 & -3 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 2a-3c & 2b-3d \\ 0 & 0 \end{bmatrix}$. Since the (2, 2)-entry of AC is not 1, AC can never equal I for any choice of the matrix C .

18. a) Find the inverse of the matrix $A = \begin{bmatrix} 1 & 4 & -1 \\ 2 & 7 & 1 \\ 1 & 3 & 0 \end{bmatrix}$ using Gauss-Jordan method.

$$A^{-1} = \begin{bmatrix} -\frac{3}{2} & -\frac{1}{2} & \frac{11}{2} \\ \frac{1}{2} & \frac{1}{2} & -\frac{3}{2} \\ -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \end{bmatrix} = \frac{1}{2} \begin{bmatrix} -3 & -1 & 11 \\ 1 & 1 & -3 \\ -1 & 1 & -1 \end{bmatrix}$$

b) Show that the matrix $A = \begin{bmatrix} 1 & 3 & -2 \\ 1 & 2 & 0 \\ 2 & 8 & -8 \end{bmatrix}$ is not invertible.

19. Let $A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 1 & 0 \\ 3 & 2 & 1 & 0 \\ 4 & 3 & 2 & 1 \end{bmatrix}$ and $b = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$.

(a) Use Gauss-Jordan elimination to compute the inverse of the matrix A.

Ans:

$$A^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & -1 & 1 & 0 \\ -1 & -2 & -1 & 0 \\ 1 & -1 & -1 & 1 \end{bmatrix}$$

(b) Use the answer in part (a) to solve the system $Ax = b$.

Ans:

$$\mathbf{x} = \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix}$$

20. Compute the determinant of

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 0 & 0 \\ 0 & 7 & 8 & 0 \\ 0 & 0 & 9 & 10 \end{bmatrix}.$$

(Ans: -530)

21. Suppose the 4 by 4 matrix M has four equal rows all containing a, b, c, d . Find determinant of this matrix

$$\begin{vmatrix} 1+a & b & c & d \\ a & 1+b & c & d \\ a & b & 1+c & d \\ a & b & c & 1+d \end{vmatrix}$$

(Ans: $1+a+b+c+d$)