Solution: Assignment 3

1. Use the method of direct proof to show that: if y is an integer, $y^2 - y$ is an even number.
   Hint: Use method of proof by cases.

   Answer:
   Let y be an integer. Then y can be either even or odd. Notice that

   $$y^2 - y = y(y-1)$$

   - **Case 1:** When y is even, we can write $y = 2k$ for some integer k. That is, we can write $y^2 - y$ as
   $$y^2 - y = y(y-1) = 2\underbrace{k(y-1)}_{\in\mathbb{Z}}.$$
   Since $k, y-1$ are integers, then the product $k(y-1)$ is also an integer.
   Hence $y^2 - y = 2\underbrace{k(y-1)}_{\in\mathbb{Z}}$ is an even number.

   - **Case 2:** When y is odd, we can write $y = 2r+1$ for some integer r. That is, we can write $y^2 - y$ as
   $$y^2 - y = (y-1)y = (2r+1-1)(2r+1) = 2r(2r+1) = 2\underbrace{r(2r+1)}_{\in\mathbb{Z}}.$$
   Since r is an integer, then the product $r(2r+1)$ is also an integer.
   Hence $y^2 - y = 2\underbrace{r(2r+1)}_{\in\mathbb{Z}}$ is an even number.

   Therefore, from **Case 1** and **Case 2** , $y^2 - y$ is even for any integer y. ■

2. Show that the equality $\lfloor x - y \rfloor = \lfloor x \rfloor - \lfloor y \rfloor$ is **not** valid for all real numbers x and y.
   Hint: Give a counterexample.

   Answer:
   Let $x = -0.7$ and $y = -0.5 \in \mathbb{R}$.
   Then $\lfloor x - y \rfloor = \lfloor -0.7 - (-0.5) \rfloor = \lfloor -0.2 \rfloor = -1$. However, $\lfloor x \rfloor = \lfloor -0.7 \rfloor = -1$ and $\lfloor y \rfloor = \lfloor -0.5 \rfloor = -1$. So $\lfloor x \rfloor - \lfloor y \rfloor = -1 - (-1) = 0$. That is, $\lfloor x - y \rfloor \neq \lfloor x \rfloor - \lfloor y \rfloor$ when $x = -0.7$ and $y = -0.5$. Hence, the statement $\lfloor x - y \rfloor = \lfloor x \rfloor - \lfloor y \rfloor$ is **not** valid for all real numbers. ■

   Note that it is also possible to use different values of x and y.

3. Use the method of constructive proof to show that:
   if r and s are two real numbers with $r < s$ then there exists a real number x such that $r < x < s$.

   Answer: Constructive proof
   Let $r, s \in \mathbb{R}$ such that $r < s$. Let
   $$x = \frac{r+s}{2}.$$

We will show that for this particular x has the value between the r and s.

$$\begin{aligned} r &< s \\ r+r &< s+r \\ \underbrace{\frac{r+r}{2}}_{=r} &< \underbrace{\frac{s+r}{2}}_{=x} \quad \Rightarrow \quad r < x \end{aligned}$$

$$\begin{aligned} r &< s \\ r+s &< s+s \\ \underbrace{\frac{r+s}{2}}_{=x} &< \underbrace{\frac{s+s}{2}}_{=s} \quad \Rightarrow \quad x < s \end{aligned}$$

That is, for any given r and s, we can always find $x = \frac{r+s}{2}$ such that $r < x < s$.

Note that it is also possible to use a different value of x.

4. (Optional) Prove by contradiction that the difference of any rational number and any irrational number is irrational.

Answer:

Proof by contradiction:

(*) Let r be a rational number and s be an irrational number.

Suppose not. I.e. suppose that the difference of r and s is a rational number. Then we would have

$$r = \frac{a}{b} \tag{1}$$

for some $a, b \in \mathbb{Z}$, $b \neq 0$ and

$$r - s = \frac{c}{d} \tag{2}$$

for some $c, d \in \mathbb{Z}$, $d \neq 0$. From (1) and (2), we have

$$\begin{aligned} r - s &= \frac{c}{d} \\ \frac{a}{b} - s &= \frac{c}{d} \\ s &= \frac{a}{b} - \frac{c}{d} = \frac{ad - bc}{bd}, \end{aligned}$$

which implies that s is a rational number (because $ad - bc$, $bd \in \mathbb{Z}$ and $bd \neq 0$) and this contradicts to the fact that s is an irrational number from (*). Hence, if r is a rational number and s is an irrational number, we must have that "the difference of r and s is an irrational number." ∎

5. For each positive integer n let $P(n)$ be the proposition $4^n - 1$ is divisible by 3.

   (a) Write $P(1)$: Is $P(1)$ true?

   (b) Write $P(k)$:

   (c) Write $P(k+1)$:

   (d) In a proof by mathematical induction that this divisibility property holds for all integers $n \geq 1$; what must be shown in the induction step?

   Answer:

   (a) $P(1)$: $4^1 - 1$ is divisible by 3.
   $P(1)$ is true because $4^1 - 1 = 4 - 1 = 3$ which is divisible by 3.

   (b) $P(k)$: $4^k - 1$ is divisible by 3.

   (c) $P(k+1)$: $4^{k+1} - 1$ is divisible by 3.

   (d) In a proof by mathematical induction that $P(n)$ holds for all integers $n \geq 1$, in the induction step, it must be shown that if $P(k)$ is true, then $P(k+1)$ is also true, for any integer $k \geq 1$. I.e., it must be shown that if $4^k - 1$ is divisible by 3, then $4^{k+1} - 1$ is also divisible by 3, for any integer $k \geq 1$.

6. Use mathematical induction proof to show that

$$2^n < n!$$

for all positive integer $n > 3$.

Answer:

Proof by mathematical induction:

Let $P(n)$ be the statement $2^n < n!$. We want to prove that $P(n)$ is true for all integer $n > 3$. Note that for $n \in \mathbb{Z}$, $n > 3$ is equivalent to $n \geq 4$ and we therefore have to use $n = 4$ in the basis step.

(I) **Basis step:** Show that $P(4)$ is true.
$P(4)$: $2^4 < 4!$.
Since $2^4 = 16$ and $4! = 1 \cdot 2 \cdot 3 \cdot 4 = 24$. Hence $2^4 < 4!$ and $P(4)$ is true.

(II)**Inductive step:** Show that if $P(k)$ is true, then $P(k+1)$ is also true, for any integer $k \geq 4$.

Assume that $P(k) : 2^k < k!$ is true. ——————————($\bigstar$) "inductive hypothesis"
We want to show that $P(k+1) : 2^{k+1} < (k+1)!$ is true. Consider

$$\begin{aligned} 2^{k+1} &= 2^k \cdot 2 \\ &< k! \cdot 2 && \text{by } (\bigstar) \text{ "inductive hypothesis"} \\ &< k! \cdot (k+1) && \text{since } 2 < k+1 \text{ for } k \geq 4 \\ &= (k+1)! \end{aligned}$$

and therefore $P(k+1)$ is true. Note that we have used $k!(k+1) = \underbrace{1 \cdot 2 \cdot 3 \cdots k}_{k!} \cdot (k+1) = (k+1)!$.

From (I) basis step and (II) inductive step, $P(n)$ is true for all $n \geq 4$ by mathematical induction proof. ■

7. (Optional) A sequence $a_1, a_2, \ldots$ is defined recursively by

$$a_1 = 3, \quad a_i = 7a_{i-1} \quad \text{for} \quad i \geq 2.$$

Show that

$$a_n = 3 \cdot 7^{n-1} \quad \text{for} \quad n \geq 1.$$

Answer:

Proof by mathematical induction:

From the given definition :

$$a_1 = 3, \quad a_i = 7a_{i-1} \quad \text{for} \quad i \geq 2. \qquad ---------------\circledast$$

Let $P(n)$ be the statement $a_n = 3 \cdot 7^{n-1}$.

We want to prove that $P(n)$ is true for all integer $n \geq 1$.

(I) **Basis step:** Show that $P(1)$ is true.

$P(1)$: $a_1 = 3 \cdot 7^{1-1}$.

Since $a_1 = 3 \cdot 7^{1-1} = 3 \cdot 7^0 = 3 \cdot 1 = 3$, which is the same as $a_1 = 3$ from the definition $\circledast$.

Hence $P(1)$ is true.

(II)**Inductive step:** Show that if $P(k)$ is true, then $P(k+1)$ is also true, for any integer $k \geq 1$.

Assume that $P(k) : a_k = 3 \cdot 7^{k-1}$ is true. ——————($\bigstar$) "inductive hypothesis"

We want to show that $P(k+1) : a_{k+1} = 3 \cdot 7^{(k+1)-1}$, or $a_{k+1} = 3 \cdot 7^k$ is true. Consider from the definition $\circledast$

$$\begin{aligned} a_{k+1} &= 7a_{[(k+1)-1]} \\ &= 7\ a_k \\ &= 7\ [3 \cdot 7^{k-1}] \qquad \text{by } (\bigstar) \text{ “inductive hypothesis : } a_k = 3 \cdot 7^{k-1} \text{ ”} \\ &= 3 \cdot 7^k \end{aligned}$$

and therefore $P(k+1)$ is true.

From (I) basis step and (II) inductive step, $P(n)$ is true for all $n \geq 1$ by mathematical induction proof. ■