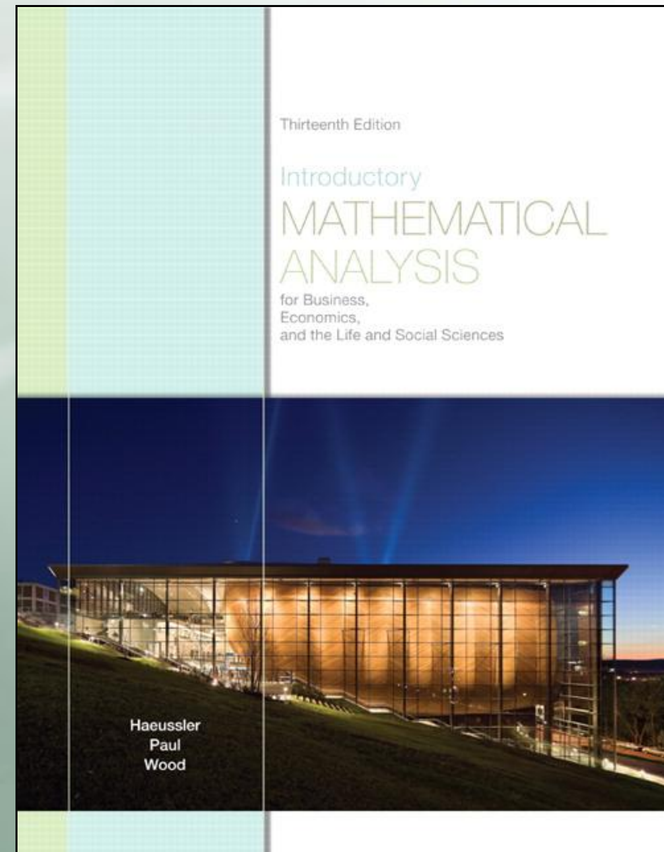
INTRODUCTORY MATHEMATICAL ANALYSIS

For Business, Economics, and the Life and Social Sciences

Chapter 2
Functions and Graphs



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Chapter Objectives

- To understand what functions and domains are.
- To introduce different types of functions.
- To introduce addition, subtraction, multiplication, division, and multiplication by a constant.
- To introduce inverse functions and properties.
- To graph equations and functions.
- To study symmetry about the *x*- and *y*-axis.
- To be familiar with shapes of the graphs of six basic functions.

Chapter Outline

2.1) Functions

2.2) Special Functions

2.3) Combinations of Functions

2.4) Inverse Functions

2.5) Graphs in Rectangular Coordinates

2.6) Symmetry

2.7) Translations and Reflections

2.1 Functions

- A **function** assigns each input number to one output number.
- The set of all input numbers is the **domain** of the function.
- The set of all output numbers is the **range**.

Equality of Functions

- Two functions f and g are equal ($f = g$):

1. Domain of f = domain of g;
2. $f(x) = g(x)$.

Example 1 – Determining Equality of Functions

Determine which of the following functions are equal.

$$\text{a. } f(x) = \frac{(x+2)(x-1)}{(x-1)}$$

$$\text{b. } g(x) = x + 2$$

$$\text{c. } h(x) = \begin{cases} x + 2 \text{ if } x \neq 1 \\ 0 \text{ if } x = 1 \end{cases}$$

$$\text{d. } k(x) = \begin{cases} x + 2 \text{ if } x \neq 1 \\ 3 \text{ if } x = 1 \end{cases}$$

Example 1 – Determining Equality of Functions

Solution:

Example 3 – Finding Domain and Function Values

Let $g(x) = 3x^2 - x + 5$. Any real number can be used for *x*, so the domain of *g* is all real numbers.

a. Find g(z).

Solution:

b. Find g(r²).

Solution:

c. *Find g(x + h)*.

Solution:

Example 5 – Demand Function

*Suppose that the equation $p = 100/q$ describes the relationship between the price per unit p of a certain product and the number of units q of the product that consumers will buy (that is, demand) per week at the stated price. Write the **demand function**.*

Solution:

2.2 Special Functions

- We begin with *constant function*.

Example 1 – Constant Function

Let *h(x)* = 2. The domain of *h* is all real numbers.

$$h(10) = 2 \qquad h(-387) = 2 \qquad h(x+3) = 2$$

A function of the form *h(x)* = *c*, where *c* = *constant*, is a **constant function.**

Example 3 – Rational Functions

a. $f(x)=\dfrac{x^2-6x}{x+5}$ is a rational function, since the numerator and denominator are both polynomials.

b. $g(x)=2x+3$ is a rational function, since $2x+3=\dfrac{2x+3}{1}$.

Example 5 – Absolute-Value Function

Absolute-value function is defined as $|x|$, e.g.

$$|x|=\begin{cases} x & \text{if } x \le 0 \\ -x & \text{if } x < 0 \end{cases}$$

Example 7 – Genetics

Two black pigs are bred and produce exactly five offspring. It can be shown that the probability P that exactly r of the offspring will be brown and the others black is a function of r ,

$$P(r) = \frac{5!\left(\frac{1}{4}\right)^{r}\left(\frac{3}{4}\right)^{5-r}}{r!\left(5-r\right)!} \qquad r = 0, 1, 2, \ldots, 5$$

On the right side, P represents the function rule. On the left side, P represents the dependent variable. The domain of P is all integers from 0 *to* 5, *inclusive. Find the probability that exactly three guinea pigs will be brown*.

Example 7 – Genetic

Solution:

2.3 Combinations of Functions

- We define the operations of function as:

$$(f+g)(x) = f(x) + g(x)$$
$$(f-g)(x) = f(x) - g(x)$$
$$(fg)(x) = f(x).g(x)$$
$$\frac{f}{g}(x) = \frac{f(x)}{g(x)} \quad \text{for } g(x) \neq 0$$

Example 1 – Combining Functions

If $f(x) = 3x - 1$ *and* $g(x) = x^2 + 3x$*, find*

a. $(f+g)(x)$
b. $(f-g)(x)$
c. $(fg)(x)$
d. $\dfrac{f}{g}(x)$
e. $(\frac{1}{2} f)(x)$

Solution:

Composition

- Composite of f with g is defined by $(f \circ g)(x) = f(g(x))$

Example 3 – Composition

If $F(p) = p^2 + 4p - 3, G(p) = 2p + 1,$ and $H(p) = |p|,$ find

a. $F(G(p))$

b. $F(G(H(p)))$

c. $G(F(1))$

Solution:

2.4 Inverse Functions

- An inverse function is defined as $f(f^{-1}(x)) = x = f^{-1}(f(x))$

Example 1 – Inverses of Linear Functions

Show that a linear function is one-to-one. Find the inverse of f(x) = ax + b and show that it is also linear.

Solution:

Example 3 – Inverses Used to Solve Equations

Many equations take the form $f(x) = 0$, *where f is a function. If f is a one-to-one function, then the equation has* $x = f^{-1}(0)$ *as its unique solution.*

Solution:

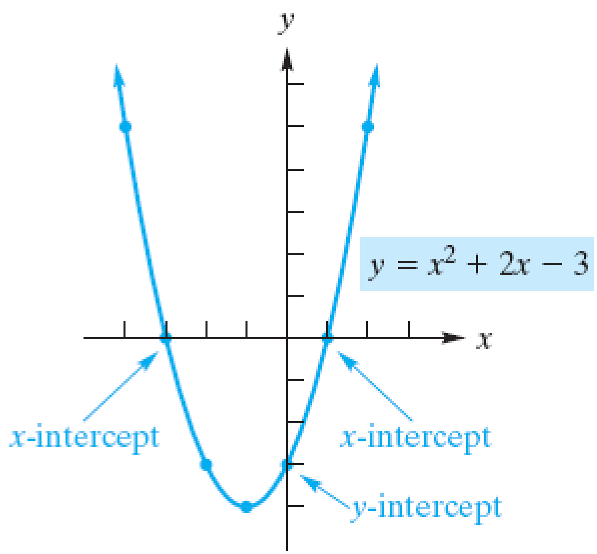
Example 5 – Finding the Inverse of a Function

To find the inverse of a one-to-one function f, solve the equation $y = f(x)$ for x in terms of y obtaining $x = g(y)$. Then $f^{-1}(x)=g(x)$. To illustrate, find $f^{-1}(x)$ if $f(x)=(x-1)^2$, for $x \geq 1$.

Solution:

2.5 Graphs in Rectangular Coordinates

- The **rectangular coordinate system** provides a geometric way to graph equations in two variables.
- An **x-intercept** is a point where the graph intersects the *x*-axis. **Y-intercept** is vice versa.



Example 1 – Intercepts and Graph

Find the x- and y-intercepts of the graph of $y = 2x + 3$, *and sketch the graph*.

Solution:

Example 3 – Intercepts and Graph

Determine the intercepts of the graph of $x = 3$, *and sketch the graph*.

Solution:

Example 7 – Graph of a Case-Defined Function

Graph the case-defined function

$$f(x) = \begin{cases} x & \text{if } 0 \le x < 3 \\ x-1 & \text{if } 3 \le x \le 5 \\ 4 & \text{if } 5 < x \le 7 \end{cases}$$

Solution:

2.6 Symmetry

- A graph is **symmetric about the y-axis** when (-*a*, *b*) lies on the graph when (*a*, *b*) does.

Example 1 – y-Axis Symmetry

Use the preceding definition to show that the graph of $y = x^2$ is symmetric about the y-axis.

Solution:

- Graph is **symmetric about the x-axis** when (x, -y) lies on the graph when (x, y) does.
- Graph is **symmetric about the origin** when ($-x$,$-y$) lies on the graph when (x, y) does.
- Summary:

Symmetry about x-axis	Replace y by $-y$ in given equation. Symmetric if equivalent equation is obtained.
Symmetry about y-axis	Replace x by $-x$ in given equation. Symmetric if equivalent equation is obtained.
Symmetry about origin	Replace x by $-x$ and y by $-y$ in given equation. Symmetric if equivalent equation is obtained.

Example 3 – Graphing with Intercepts and Symmetry

Test $y = f(x) = 1 - x^4$ for symmetry about the x-axis, the y-axis, and the origin. Then find the intercepts and sketch the graph.

Example 3 – Graphing with Intercepts and Symmetry

Solution:

- A graph is **symmetric about the $y = x$** when (b, a) and (a, b).

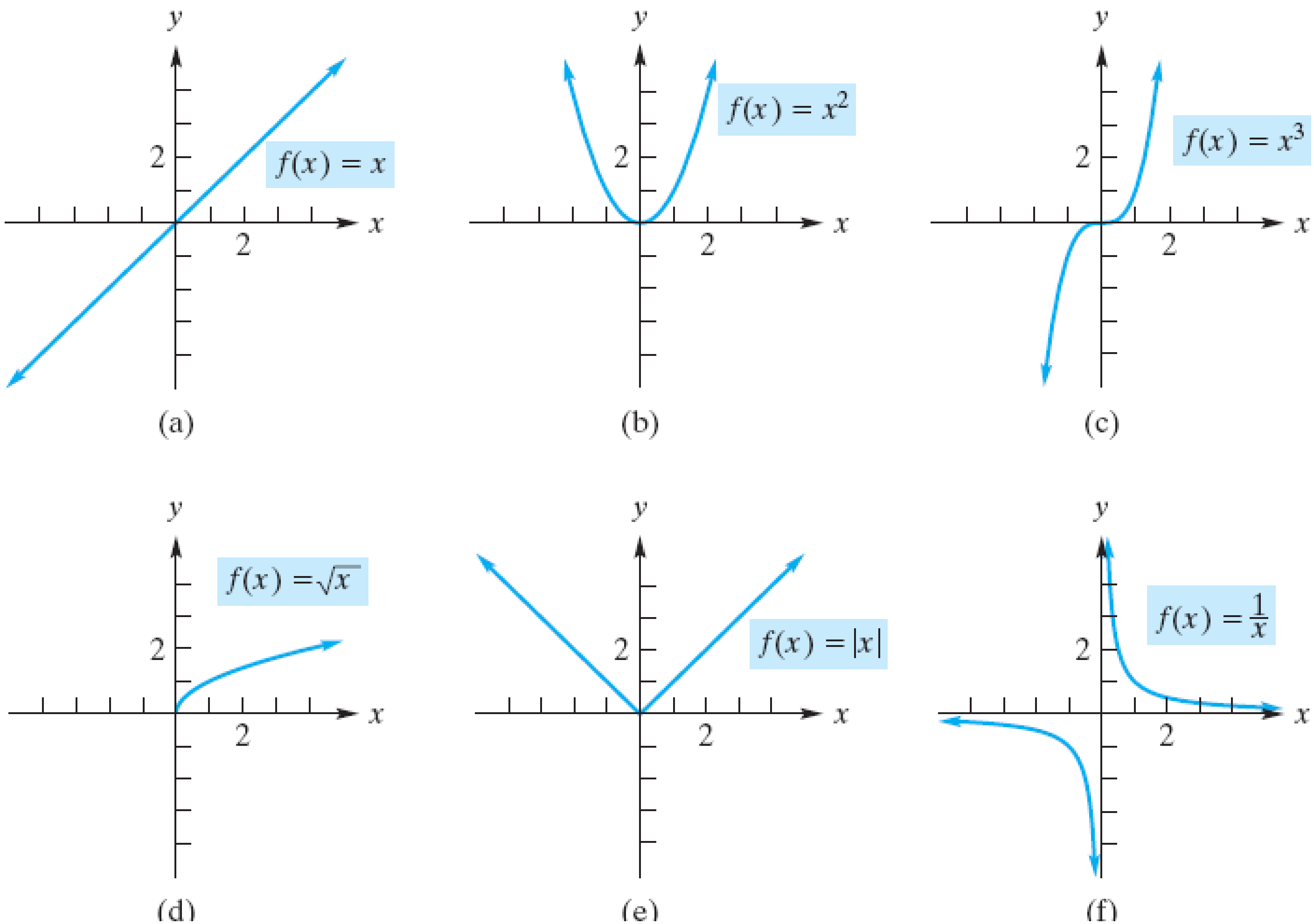
Example 5 – Symmetry about the Line $y = x$

Show that $x^2 + y^2 = 1$ *is symmetric about the line* $y = x$.

Solution:

2.7 Translations and Reflections

- 6 frequently used functions:



- Basic types of transformation:

Equation	How to Transform Graph of $y = f(x)$ to Obtain Graph of Equation
$y = f(x) + c$	shift c units upward
$y = f(x) - c$	shift c units downward
$y = f(x - c)$	shift c units to right
$y = f(x + c)$	shift c units to left
$y = -f(x)$	reflect about x-axis
$y = f(-x)$	reflect about y-axis
$y = cf(x) \quad c > 1$	vertically stretch away from x-axis by a factor of c
$y = cf(x) \quad c < 1$	vertically shrink toward x-axis by a factor of c

Example 1 – Horizontal Translation

Sketch the graph of $y = (x - 1)^3$.

Solution: