OLS Matrix Approach

17 สิงหาคม 2560 8:02

Enrollment Key $\boxed{1341}$
EE435

$$Y_i = \beta_1 + \beta_2 X_{2i} + \cdots + \beta_k X_{ki} + u_i \qquad i = 1, 2, \ldots, n$$

$$\begin{array}{l} i=1 \\ i=2 \\ \vdots \\ i=n \end{array} \quad \underset{n\times 1}{\begin{bmatrix} Y_1 \\ Y_2 \\ \vdots \\ Y_n \end{bmatrix}} \begin{array}{c} = \\ = \\ \\ = \end{array} \underset{n\times 1}{\begin{bmatrix} \beta_1 + \beta_2 X_{21} + \cdots + \beta_k X_{k1} \\ \beta_1 + \beta_2 X_{22} + \cdots + \beta_k X_{k2} \\ \vdots \\ \beta_1 + \beta_2 X_{2n} + \cdots + \beta_k X_{kn} \end{bmatrix}} + \underset{n\times 1}{\begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix}}$$

$$\underset{n\times 1}{\begin{bmatrix} Y_1 \\ Y_2 \\ \vdots \\ Y_n \end{bmatrix}} = \underset{n\times k}{\begin{bmatrix} 1 & X_{21} & \cdots & X_{k1} \\ 1 & X_{22} & \cdots & X_{k2} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & X_{2n} & \cdots & X_{kn} \end{bmatrix}} \cdot \underset{k\times 1}{\begin{bmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_k \end{bmatrix}} + \underset{n\times 1}{\begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix}}$$

$$\boxed{\underset{n\times 1}{Y} = \underset{n\times k}{X}\underset{k\times 1}{\beta} + \underset{n\times 1}{u}} \qquad u = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix}$$

$$u' = [u_1 \; u_2 \cdots u_n]$$

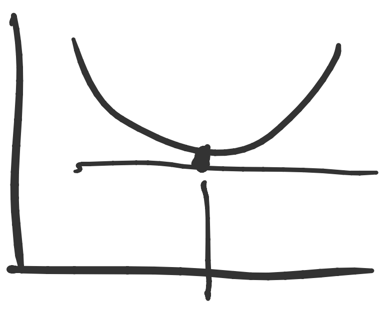
$$\hat{u}'\hat{u} = \underset{1\times n}{[\hat{u}_1 \; \hat{u}_2 \cdots \hat{u}_n]} \underset{n\times 1}{\begin{bmatrix} \hat{u}_1 \\ \hat{u}_2 \\ \vdots \\ \hat{u}_n \end{bmatrix}} = \hat{u}_1^2 + \hat{u}_2^2 + \cdots + \hat{u}_n^2 = \sum_{i=1}^{n} \hat{u}_i^2$$

$$Y = X\hat{\beta} + \hat{u} \Rightarrow \underline{\hat{u} = (Y - X\hat{\beta})}$$

$$\hat{u}'\hat{u} = (Y - X\hat{\beta})'(Y - X\hat{\beta})$$

$$\hat{u}'\hat{u} = Y'Y - \underbrace{\underset{1\times n}{Y'}\underset{n\times k}{X}\underset{k\times 1}{\hat{\beta}}}_{1\times 1} - \underbrace{\underset{1\times k}{\hat{\beta}'}\underset{k\times n}{X'}\underset{n\times 1}{Y}}_{1\times 1} + \hat{\beta}'X'X\hat{\beta}$$

$$\frac{\partial \hat{u}'\hat{u}}{\partial \hat{\beta}} = \frac{\partial Y'Y}{\partial \hat{\beta}} - \frac{\partial\, 2\hat{\beta}'X'Y}{\partial \hat{\beta}} + \frac{\partial\, \hat{\beta}'X'X\hat{\beta}}{\partial \hat{\beta}} = 0$$



$$\begin{array}{c} \| \\ 0 \end{array} \quad -2X'Y + 2X'X\hat{\beta} = 0$$

$$\ddot{0} \quad -2X'Y + 2X'X\hat{\beta} = 0$$

$$\cancel{2}X'X\hat{\beta} = \cancel{2}X'Y$$

$$\cancel{(X'X)^{-1}}\cancel{X'X}\hat{\beta} = (X'X)^{-1}X'Y$$

$$\underset{k\times 1}{\hat{\beta}} = \underset{k\times n}{(X'X)^{-1}}\ \underset{n\times k}{X'}\ \underset{k\times n}{}\ \underset{n\times 1}{Y}$$

$$E(\underset{n\times 1}{u}\cdot\underset{1\times n}{u'}) = \begin{bmatrix} E(u_1u_1) & E(u_1u_2) & \cdots & E(u_1u_n) \\ E(u_2u_1) & E(u_2u_2) & \cdots & E(u_2u_n) \\ \vdots & & & \\ E(u_nu_1) & E(u_nu_2) & \cdots & E(u_nu_n) \end{bmatrix} = \begin{bmatrix} \sigma_1^2 & \sigma_{12} & \cdots & \sigma_{1n} \\ \sigma_{21} & \sigma_2^2 & \cdots & \sigma_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \sigma_{n1} & \sigma_{n2} & \cdots & \sigma_n^2 \end{bmatrix}$$

$$\begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix}\begin{bmatrix} u_1 & u_2 & \cdots & u_n \end{bmatrix}$$

$$E\left[(x_i - E(x_i))^2\right] = var(x_i)$$

$$E\left[(u_i - \underset{\overset{\shortparallel}{0}}{E(u_i)})^2\right] = E(u_i^2) \underset{i=1}{=} E(u_1u_1) = var(u_1) = \sigma_1^2$$

$$E\left[(x_i - E(x_i))(y_i - E(y_i))\right] = cov.(x_i y_i)$$

$$E\left[(u_1 - E(u_1))(u_2 - E(u_2))\right] = E(u_1u_2) = cov(u_1,u_2) = \sigma_{12}$$

$$\underset{\overset{\shortparallel}{\Sigma}}{E(u\cdot u')} = \begin{bmatrix} \sigma_1^2 & \sigma_{12} & \cdots & \sigma_{1n} \\ \sigma_{21} & \sigma_2^2 & \cdots & \sigma_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \sigma_{n1} & \sigma_{n2} & \cdots & \sigma_n^2 \end{bmatrix} = \underset{n\times n}{\begin{bmatrix} \sigma^2 & 0 & \cdots & 0 \\ 0 & \sigma^2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \sigma^2 \end{bmatrix}} = \sigma^2 \underset{n\times n}{I}$$

$$=0$$

Assume $\boxed{\text{Homo}}$ or Non auto

Cross-section Time-Series

$$\sigma_1^2 = \sigma_2^2 \cdots = \sigma_n^2 = \sigma^2 \qquad cov(u_i u_j) = 0$$

constant var.

Generalized Least Squares (GLS) Relax - (~~Homo~~ ~~Non auto~~)

OLS

(Homo & Non auto)

$$\hat{\beta} = (X'X)^{-1}X'Y$$

$$\boxed{\underset{n\times n}{\Sigma} = \sigma^2 \underset{n\times n}{I}}$$

(Homo & Non auto)

$n \times n$ $n \times n$

$$\hat{\beta}_{GLS} = (X'\Sigma^{-1}X)^{-1} X'\Sigma^{-1} Y$$

$$\hat{\beta}_{OLS} = (X'(\sigma^2 I)^{-1} X)^{-1} X'(\sigma^2 I)^{-1} Y$$

$$= \cancel{\sigma^2} \cdot \frac{1}{\cancel{\sigma^2}} (X' I^{-1} X)^{-1} X' I^{-1} Y$$

$$= (X'X)^{-1} X'Y$$

Heteroscedasticity

$$\boxed{\hat{\Sigma} = \begin{bmatrix} \hat{\sigma}_1^2 & 0 & \cdots & 0 \\ 0 & \hat{\sigma}_2^2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \hat{\sigma}_n^2 \end{bmatrix}}$$

$$\hat{\beta}_{FGLS} = (X'\hat{\Sigma}^{-1}X)^{-1} X'\hat{\Sigma}^{-1} Y$$

Feasible GLS