Solution: Exercise 1 (Part 3)

1. Determine whether the statement forms are logically equivalent. In each case, construct a truth table to justify your Solution.

   (a) $(p \to q) \to (q \to p)$ and $(p \vee q) \to (p \wedge q)$

   (b) $(p \vee q) \wedge \sim (p \wedge q)$ and $p \leftrightarrow q$

   (c) $(p \leftrightarrow q) \leftrightarrow r$ and $p \leftrightarrow (q \leftrightarrow r)$

   Solution:

   (a) $(p \to q) \to (q \to p)$ and $(p \vee q) \to (p \wedge q)$

   Truth table:

p	q	$p \to q$	$q \to p$	$p \vee q$	$p \wedge q$	$(p \to q) \to (q \to p)$	$(p \vee q) \to (p \wedge q)$
T	T	T	T	T	T	T	T
T	F	F	T	T	F	T	F
F	T	T	F	T	F	F	F
F	F	T	T	F	F	T	T

   Since rows 3 and 4 of the truth table for $(p \to q) \to (q \to p)$ and $(p \vee q) \to (p \wedge q)$ have different truth values, then these statement forms are not logically equivalent. ■

   (b) $(p \vee q) \wedge \sim (p \wedge q)$ and $p \leftrightarrow q$

   Truth table:

p	q	$p \vee q$	$p \wedge q$	$\sim (p \wedge q)$	$(p \vee q) \wedge \sim (p \wedge q)$	$p \leftrightarrow q$
T	T	T	T	F	F	T
T	F	T	F	T	T	F
F	T	T	F	T	T	F
F	F	F	F	T	F	T

   Since all 4 rows of the truth table for $(p \vee q) \wedge \sim (p \wedge q)$ and $p \leftrightarrow q$ have different truth values, then these statement forms are not logically equivalent. ■

   (c) Truth table for $(p \leftrightarrow q) \leftrightarrow r$ and $p \leftrightarrow (q \leftrightarrow r)$.

p	q	r	$p \leftrightarrow q$	$q \leftrightarrow r$	$(p \leftrightarrow q) \leftrightarrow r$	$(p \leftrightarrow q) \leftrightarrow r$
T	T	T	T	T	T	T
T	T	F	T	F	F	F
T	F	T	F	F	F	F
T	F	F	F	T	T	T
F	T	T	F	T	F	F
F	T	F	F	F	T	T
F	F	T	T	F	T	T
F	F	F	T	T	F	F

   Since all 8 rows (all possible cases for truth values of p, q, r) in the truth table for $(p \leftrightarrow q) \leftrightarrow r$ and $p \leftrightarrow (q \leftrightarrow r)$ have the same truth values, then these statement forms are logically equivalent. ■

2. Define the truth table for a connective operator $\oplus$ as follow.

p	q	$p \oplus q$
T	T	F
T	F	T
F	T	T
F	F	F

Construct a truth table for each of the following statement forms.

(a) $p \oplus p$ (b) $(p \oplus p) \oplus p$ (c) $(p \oplus p) \oplus (p \oplus p)$

<u>Solution:</u> The truth table for (a)-(c) is shown below.

p	p	(a) $p \oplus p$	(b) $(p \oplus p) \oplus p$	(c) $(p \oplus p) \oplus (p \oplus p)$
T	T	F	T	F
F	F	F	F	F

3. Determine whether or not each of the following statements is a tautology, a contradiction, or neither of them.

(a) $(p \leftrightarrow q) \to p \wedge q$ (b)$\sim (p \leftrightarrow q) \to \sim (p \wedge q)$

<u>Solution:</u> Truth table for $(p \leftrightarrow q) \to p \wedge q$

p	q	$p \leftrightarrow q$	$p \wedge q$	$(p \leftrightarrow q) \to p \wedge q$
T	T	T	T	T
T	F	F	F	T
F	T	F	F	T
F	F	T	F	F

From the truth table, since there are both true and false rows for $(p \leftrightarrow q) \to p \wedge q$, this statement form is neither a tautology nor a contradiction. ■

(b) Truth table for $\sim (p \leftrightarrow q) \to \sim (p \wedge q)$

p	q	$\sim (p \leftrightarrow q)$	$\sim (p \wedge q)$	$\sim (p \leftrightarrow q) \to \sim (p \wedge q)$
T	T	F	F	T
T	F	T	T	T
F	T	T	T	T
F	F	F	T	T

Since all the rows (all possible cases for the truth values for p and q) in the truth table for $p \wedge q \to (p \to q)$ have the **true** truth value, then this statement form is a **tautology**. ■

4. Let p, q and r be statements such that $(p \leftrightarrow q) \vee (q \rightarrow r)$ is **false**. Determine the truth value of $(p \vee q \vee r) \wedge (\sim p \vee \sim q \vee \sim r)$. Explain your Solution.

   Solution: **True** Since $(p \leftrightarrow q) \vee (q \rightarrow r)$ is false, then both $(p \leftrightarrow q)$ and $(q \rightarrow r)$ are false.

   (i) To have $(p \leftrightarrow q)$ false, p and q must have opposite truth vales.

   (ii) To have $(q \rightarrow r)$ false, q must be true and r must be false.

   From (i) and (ii), since q is true, p must be false.

p	q	r	$p \vee q \vee r$	$\sim p \vee \sim q \vee \sim r$	$(p \vee q \vee r) \wedge (\sim p \vee \sim q \vee \sim r)$
F	T	F	T	T	T

   Note that there are other approaches to complete this problem. ■

5. Consider the following statement.

   If I do not go to class and I get an A for that class, then I am smart.

   (a) Write the **negation** of the above statement.

   (b) Write the **contrapositive**, **inverse**, and **converse** of the above statement.

   Solution:
   Let p be “I do not go to class ,”
   q be “I get an A for that class,”
   r be “I am smart.”
   Then the statement can be written as

$$(p \wedge q) \rightarrow r.$$

   (a) The **negation :**

$$\sim [(p \wedge q) \rightarrow r] \equiv (p \wedge q) \wedge \sim r$$

   “I do not go to class and I get an A for that class , but(and) I am not smart”

   (b) **Contrapositive:**

$$\sim r \rightarrow \sim (p \wedge q) \equiv \sim r \rightarrow (\sim p \vee \sim q)$$

   “If I am not smart, then I go to class **or** *I do not get an A for that class.”*

   Inverse:

$$\sim (p \wedge q) \rightarrow \sim r \equiv (\sim p \vee \sim q) \rightarrow \sim r$$

   “If I go to class **or** *I do not get an A for that class, then I am not smart.”*

   Converse

$$r \rightarrow (p \wedge q)$$

   “If I am smart, then I do not go to class **and** *I get an A for that class.”*

■

6. Use truth tables to determine whether the argument forms are valid. Indicate which columns represent the premises and which represent the conclusion, and include a sentence explaining how the truth table supports your Solution.

(a)
$$\begin{array}{rl} & p \to \sim q \\ & \sim q \to p \\ \therefore & p \vee \sim q \end{array}$$

(b)
If I skip classes and I pass these classes, then I read the text books.
I do not read the text books or I skip classes.
∴ I do not pass these classes.

(c) (Optional)
If I skip classes and I pass these classes, then I read the text books.
I do not read the text books.
I skip classes.
∴ I do not pass these classes.

(d) (Optional)
If I skip classes and I pass these classes, then I read the text books.
I do not read the text books.
I pass these classes
∴ I am smart.

<u>Solution:</u>

(a)
$$\begin{array}{rl} & p \to \sim q \\ & \sim q \to p \\ \therefore & p \vee \sim q \end{array}$$

Truth table:

			Premises		Conclusion	
p	q	$\sim q$	$p \to \sim q$	$\sim q \to p$	$p \vee \sim q$	
T	T	F	F	T		
T	F	T	$\boxed{\text{T}}$	$\boxed{\text{T}}$	T	←-- critical row
F	T	F	$\boxed{\text{T}}$	$\boxed{\text{T}}$	F	←-- critical row: This has true premises but false conclusion.
F	F	T	T	F		

Notice that columns 4 and 5 consist of the truth values of premises and rows 2 and 3 are critical rows (premises are all true). Since row 3 (which is a critical row) has true premises with false conclusion, then this argument is **invalid**. ■

(b) Invalid.

Let p be "I skip classes"; q be "I pass these classes," and r be "I read the text books."

	If I skip classes and I pass these classes, then I read the text books.		$p \wedge q \to r$
	I do not read the text books or I skip classes.		$\sim r \vee p$
∴	I do not pass these classes.	∴	$\sim q$

Truth table:

					Premises		Conclusion	
p	q	r	$\sim r$	$p \wedge q$	$p \wedge q \to r$	$\sim r \vee p$	$\sim q$	
T	T	T	F	T	$\boxed{T}$	$\boxed{T}$	$\boxed{F}$	←-- critical row
T	T	F	T	T	F	T	F	
T	F	T	F	F	$\boxed{T}$	$\boxed{T}$	$\boxed{T}$	←-- critical row
T	F	F	T	F	$\boxed{T}$	$\boxed{T}$	$\boxed{T}$	←-- critical row
F	T	T	F	F	T	F	F	
F	T	F	T	F	$\boxed{T}$	$\boxed{T}$	$\boxed{F}$	←-- critical row
F	F	T	F	F	T	F	T	
F	F	F	T	F	$\boxed{T}$	$\boxed{T}$	$\boxed{T}$	←-- critical row

Notice that columns 6 and 7 give the truth values of premises and the critical rows are rows 1,3,4, 6 and 8. Since there are some critical rows (rows 1 and 6) that have false conclusions, then this argument is **invalid**. ■

(c) Valid.
Let p be "I skip classes"; q be "I pass these classes," and r be "I read the text books."

	If I skip classes and I pass these classes, then I read the text books.	$p \wedge q \to r$
	I do not read the text books.	$\sim r$
	I skip classes.	p
$\therefore$	I do not pass these classes.	$\therefore \sim q$

Truth table:

				Premises			Conclusion	
p	q	r	$p \wedge q$	$p \wedge q \to r$	$\sim r$	p	$\sim q$	
T	T	T	T	T	F	T	F	
T	T	F	T	F	T	T	F	
T	F	T	F	T	F	T	T	
T	F	F	F	$\boxed{T}$	$\boxed{T}$	$\boxed{T}$	$\boxed{T}$	←-- critical row
F	T	T	F	T	F	F	F	
F	T	F	F	T	T	F	F	
F	F	T	F	T	F	F	T	
F	F	F	F	T	T	F	T	

Notice that columns 5, 6 and 7 give the truth values of premises and the only critical row is row 4. Since this critical row has true conclusion, then this argument is **valid**. ■

(d) Invalid.

■

7. (Optional) Consider the following premises.

(i) It is not sunny this afternoon and it is colder than yesterday.

(ii) We will go swimming only if it is sunny.

(iii) If we do not go swimming, then we will take a canoe trip.

(iv) If we take a canoe trip, then we will be home by sunset.

From the above premises (i)-(iv), does the conclusion that *we go home by sunset* make a valid argument? Explain your Solution by using rules of inferences.

Solution:

Let p be the statement "It is sunny this afternoon,"
q the statement "It is colder than yesterday,"
r the statement "We will go swimming,"
s the statement "We will take a canoe trip," and t the statement "We will be home by sunset."
Then the argument consists of the following premises and conclusion.

$$\begin{array}{ll} & \sim p \wedge q \\ & r \to p \\ & \sim r \to s \\ & s \to t \\ \therefore & t \end{array}$$

We construct an argument to show that our premises lead to the desired conclusion as follows.

(1) Simplification : premise (i)

$$\begin{array}{ll} & \sim p \wedge q \\ \therefore & \sim p \end{array}$$

(2) Modus tollens: using premise (ii) and (1)

$$\begin{array}{lll} & r \to p & \text{premise (ii)} \\ & \sim p & \text{from (1)} \\ \therefore & \sim r & \end{array}$$

(3) Modus ponens: using premise (iii) and (2)

$$\begin{array}{lll} & \sim r \to s & \text{premise (iii)} \\ & \sim r & \text{from (2)} \\ \therefore & s & \end{array}$$

(4) Modus ponens: using premise (iv) and (3)

$$\begin{array}{lll} & s \to t & \text{premise (iv)} \\ & s & \text{from (3)} \\ \therefore & t & \end{array}$$

Therefore, we can conclude that "we will be home by sunset," from rules of inference used in (1)-(4).

Note that we could have used a truth table to show that whenever each of the four hypotheses is true, the conclusion is also true. However, because we are working with five propositional variables, p, q, r, s, and t , such a truth table would have 32 rows. ■

8. (Optional) The famous detective Percule Hoirot was called in to solve a baffling murder mystery. He determined the following facts:

(a) Lord Hazelton, the murdered man, was killed by a blow on the head with a brass candlestick.

(b) Either Lady Hazelton or a maid, Sara, was in the dining room at the time of the murder.

(c) If the cook was in the kitchen at the time of the murder, then the butler killed Lord Hazelton with a fatal dose of strychnine.

(d) If Lady Hazelton was in the dining room at the time of the murder, then the chauffeur killed Lord Hazelton.

(e) If the cook was not in the kitchen at the time of the murder, then Sara was not in the dining room when the murder was committed.

(f) If Sara was in the dining room at the time the murder was committed, then the wine steward killed Lord Hazelton.

Is it possible for the detective to deduce the identity of the murderer from these facts? If so, who did murder Lord Hazelton? (Assume there was only one cause of death.) Explain your Solution by using rules of inferences.

Solution:

The chauffeur killed Lord Hazelton. One explanation:

(1) Suppose the cook was in the kitchen at the time of the murder.

(2) $\therefore$ The butler killed Lord Hazelton with strychnine. (by (c) and (1) and modus ponens)

(3) $\therefore$ We have a contradiction: Lord Hazelton was killed by strychnine and a blow on the head. (by (2) and (a))

(4) $\therefore$ The supposition that the cook was in the kitchen is false. (by the contradiction rule)

(5) $\therefore$ The cook was not in the kitchen at the time of the murder. (negation of supposition)

(6) $\therefore$ Sara was not in the dining room when the murder was committed. (by (e) and (5) and modus ponens)

(7) $\therefore$ Lady Hazelton was in the dining room when the murder was committed. (by (b) and (6) and elimination)

(8) $\therefore$ The chauffeur killed Lord Hazelton. (by (d) and (7) and modus ponens)

That is, the chauffeur murdered Lord Hazelton.

■