Quiz 2: Date: May 5, 2022 from 11.00-12.30

Question 1 (40 marks) Score......................

Consider the Muliperiod model of consumption and portfolio choice. Let an individual in this economy has the utility function as follow:

$$\max_{C_s,\omega_s,\forall t} \; E_t\left[\sum_{s=t}^{T-1}\delta^s\left(\frac{C_s^{(1-\gamma)}}{1-\gamma}\right)+\delta^T\left(\frac{W_T^{(1-\gamma)}}{1-\gamma}\right)\right]$$

Assume that there is no wage income $(y_t = 0\,\forall t)$ and a constant risk-free rate return asset , $R_{ft} = R_f$. Also asssume that n=1 and the return of a single risky asset, R_{rt},is independently and identically distributed over time. Denote the proportion of wealth invested in the risky asset at date t as ω_t.

Please read and answer the following questions carefully and completely.

Score......................

Question 1.1 (10 marks) Derive the first-order condition for the optimal consumption level and portfolio weight at date T-1, C^*_{T-1} and ω^*_{T-1}., and give an explicit expression for C^*_{T-1}

$$J(W_T, T-1) = \max_{C_S, W_S, \forall s} E_t\left[\sum_{s=t}^{T-1} \delta^s \left(\frac{C_s^{(1-\gamma)}}{1-\gamma}\right) + \left(\frac{W_T^{(1-\gamma)}}{1-\gamma}\right)\right]$$

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$$U_c(C_{T-1}, T-1) = E_{T-1}[B_W(W_T, T)R_{T-1}] \rightarrow \text{Date } T-1$$

$$\delta^{T-1}(C_{T-1}^{-\gamma}) = E_{T-1}[\delta^T(R_{T-1}) \cdot W_{T-1}^{-\gamma}]$$

$$\delta^{T-1} C_{T-1}^{-\gamma} = \delta^T \cdot S_{T-1}^{-\gamma} \cdot R_{T-1}^{1-\gamma}$$

$$C^*_{T-1} = \delta \cdot W_{T-1}^{-\gamma} \text{ ※}$$

$$J(W_{T-1}, T-1) = \max_{(T-1, (w_i, T-1))} E_{T-1}[U(C_{T-1}, T-1) + B(W_T, T)]$$

$$= \max U[C_{T-1}, T-1] + E_{T-1}[B(W_{T-1})]$$

note: $S_{T-1} = W_{T-1} + Y_{T-1} - C_{T-1}$

$$R_{T-1} = R_{f,T-1} = \sum_{i=1}^{\infty} w_{i,T-1}(R_{i,T-1} - R_{f,T-1})$$

$$\frac{\partial J}{\partial C_{T-1}} = \delta^{T-1}(1-\gamma)\frac{C_{T-1}^{-\gamma}}{1-\gamma} - E_{T-1}\left[\delta^T(1-\gamma)\frac{W_T^{-\gamma}}{1-\gamma} \cdot R_{T-1}\right] = 0$$

$$= E_{T-1}\left[\delta^T(1-\gamma)\frac{W^{-\gamma}}{1-\gamma}(R_{i,T-1} - R_{f,T-1})\right] = 0$$

$$\therefore \delta^{T-1} C_{T-1}^{-\gamma} = E_{T-1}\left[\delta^T W_T^{-\gamma}\left(R_{f,T-1} + \sum_{i=1}^{n} w_{i,T-1}(R_{i,T-1} - R_{f,T-1})\right)\right]$$

$$\delta^{T-1} C_{T-1}^{-\gamma} = R_{f,T-1} E_{T-1}[\delta^T W_T^{-\gamma}]$$

$$\max_{C_s,\omega_s,\forall_t} E_t\left[\sum_{s=t}^{T-1}\delta^s\left(\frac{C_s^{(1-\gamma)}}{1-\gamma}\right)+\delta^T\left(\frac{W_T^{(1-\gamma)}}{1-\gamma}\right)\right]$$

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Score......................

Question 1.2 (10 marks) Solve for the form of J(W_{T-1},T-1).

$$J(W_T, T-1) = \max_{Cs, Ws, \forall s} E_t\left[\sum_{s=t}^{T-1} \delta^s \left(\frac{C_s^{(1-\gamma)}}{1-\gamma}\right) + \left(\frac{W_T^{(1-\gamma)}}{1-\gamma}\right)\right]$$

Score.....................

Question 1.3 (10 marks) Derive the first-order condition for the optimal consumption level and portfolio weight at date T-2, C^*_{T-2} and ω^*_{T-2}., and give an explicit expression for C^*_{T-2}

$$E_{T-2}(B_w R_{r,T-2}) = R_f E_{T-2}(B_w)$$

$$\delta^T E_{T-2}[(S_{T-2}R_{T-2})^{\gamma-1} R_{r,T-2}] = \delta^T_{f,T-2} E_{T-2}[(S_{T-2}R_{T-2})^{\gamma-1}]$$

$$E_{T-2}[R^{\gamma-1}_{T-2} R_{r,T-1}] = R_{f,T-2} E_{T-2}[R^{\gamma-1}_{T-2}]$$

$$E[R^{\gamma-1}_{T-2} R_{r,T-2}] = R_{f,T-2} E[R^{\gamma-1}_{T-2}]$$

optimal portfolio weight is determined independently from the level of wealth or consumption. It depends only on distribution of returns on the risky asset. Since distribution is independent and identically distributed, we can replace the condition with unconditional expectations operator. All individuals with the same constant coefficient of relative risk aversion, γ chose the same portfolio proportions for risky and risk free asset.

Score.....................

Question 1.4 (10 marks) Solve for the form of J(W_{T-2},T-2).Based on the pattern for T-1 and T-2, provide expressions for the optimal consumption and portfolio weight at any date T-t, t=1,2,3,....

$$J(W_{T-2}, T-2) = \max_{c_s, w_s, \forall s} E\left[\sum_{s=t}^{T-2} \delta^s\left(\frac{C_s^{(1-r)}}{1-r}\right) + \left(\frac{W_T^{(1-r)}}{1-r}\right)\right]$$