Quiz 2: Date: May 5, 2022 from 11.00-12.30

Question 1 (40 marks) Score....................

Consider the Muliperiod model of consumption and portfolio choice. Let an individual in this economy has the utility function as follow:

$$\max_{C_s,\omega_s,\forall_t} E_t\left[\sum_{s=t}^{T-1}\delta^s\left(\frac{C_s^{(1-\gamma)}}{1-\gamma}\right)+\delta^T\left(\frac{W_T^{(1-\gamma)}}{1-\gamma}\right)\right]$$

Assume that there is no wage income $(y_t = 0\,\forall t)$ and a constant risk-free rate return asset , $R_{ft} = R_f$. Also asssume that n=1 and the return of a single risky asset, R_{rt},is independently and identically distributed over time. Denote the proportion of wealth invested in the risky asset at date t as ω_t.

Please read and answer the following questions carefully and completely.

Score....................

Question 1.1 (10 marks) Derive the first-order condition for the optimal consumption level and portfolio weight at date T-1, C^*_{T-1} and ω^*_{T-1}., and give an explicit expression for C^*_{T-1}

$$\max_{C_s,\omega_s,\forall_t} E_t\left[\sum_{s=t}^{T-1} \underbrace{\delta^s\left(\frac{C_s^{(1-\gamma)}}{1-\gamma}\right)}_{U(C_t,t)} + \underbrace{\delta^T\left(\frac{W_T^{(1-\gamma)}}{1-\gamma}\right)}_{B(W_T,T)}\right]$$

, $S_t = W_t - C_t$ and $R_t = R_f + \omega_t(R_{rt} - R_f)$

Bellman's Equation

$$J(W_t,t) = \max_{C_t,\omega_t} U(C_t,t) + E_t[J(W_{t+1},t+1)]$$

- At date T, $J(W_T,T) = E_T[B(W_T,T)] = \delta^T\left(\frac{W_T^{(1-\gamma)}}{1-\gamma}\right)$

- At date T-1,

$$J(W_{T-1},T-1) = \max_{C_{T-1},\omega_{T-1}} \left(\delta^{T-1}\frac{C_{T-1}^{(1-\gamma)}}{(1-\gamma)} + E_{T-1}\left[\overbrace{\delta^T\frac{W_T^{1-\gamma}}{1-\gamma}}^{J(W_T,T)}\right]\right)$$

Optimality conditions at T-1

$$U_c(C^*_{T-1},T-1) = E_{T-1}[J_W(W_T,T)R_{T-1}] \quad \text{--- (1)}$$

$$\delta^{T-1}C_{T-1}^{-\gamma} = E_{T-1}[\delta^T W_T^{-\gamma}R_{T-1}] = E_{T-1}[\delta^T(S_{T-1}R_{T-1})^{-\gamma}R_{T-1}]$$

$$\delta^{T-1}C_{T-1}^{-\gamma} = \delta^T S_{T-1}^{-\gamma}E_{T-1}[R_{T-1}^{1-\gamma}]$$

$$\delta^{T-1} c_{T-1}^{-\gamma} = \delta^{T}(W_{T-1}-c_{T-1})^{-\gamma} E_{T-1}\left[R_{T-1}^{1-\gamma}\right]$$

$$c_{T-1} = \left(\delta (W_{T-1}-c_{T-1})^{-\gamma} E_{T-1}\left[R_{T-1}^{1-\gamma}\right]\right)^{\frac{1}{-\gamma}}$$

$$c_{T-1} = (W_{T-1}-c_{T-1})\left(\delta E_{T-1}\left[R_{T-1}^{1-\gamma}\right]\right)^{\frac{1}{-\gamma}}$$

$$c_{T-1}\left(1+\left(\delta E_{T-1}\left[R_{T-1}^{1-\gamma}\right]\right)^{\frac{1}{-\gamma}}\right) = \left(\delta E_{T-1}\left[R_{T-1}^{1-\gamma}\right]\right)^{\frac{1}{-\gamma}} W_{T-1}$$

$$\boxed{c_{T-1}^{*} = \frac{\left(\delta E_{T-1}\left[R_{T-1}^{1-\gamma}\right]\right)^{\frac{1}{-\gamma}} W_{T-1}}{\left(1+\left(\delta E_{T-1}\left[R_{T-1}^{1-\gamma}\right]\right)^{\frac{1}{-\gamma}}\right)} = \frac{a_1}{1+a_1} W_{T-1} = c_1 W_{T-1}}$$

where $a_1 = \left(\delta E_{T-1}\left[R_{T-1}^{1-\gamma}\right]\right)^{\frac{1}{-\gamma}} = \left(\delta E\left[R_{T-1}^{1-\gamma}\right]\right)^{\frac{1}{-\gamma}}$, $c_1 = \frac{a_1}{1+a_1}$

since R_f is fixed and $R_{rt} \sim$ i.i.d.

$$E_{T-1}\left[R_{r,T-1} J_W(W_T)\right] = R_f E_{T-1}\left[J_W(W_T)\right] \quad \text{—— (2)}$$

$$E_{T-1}\left[R_{r,T-1}\, \delta^{T} W_T^{-\gamma}\right] = R_f E_{T-1}\left[\delta^{T} W_T^{-\gamma}\right]$$

$$\cancel{\delta^{T}} E_{T-1}\left[R_{r,T-1}\, \cancel{S_{T-1}^{-\gamma}} R_{T-1}^{-\gamma}\right] = \cancel{\delta^{T}} R_f E_{T-1}\left[\cancel{S_{T-1}^{-\gamma}} R_{T-1}^{-\gamma}\right]$$

$$\boxed{E\left[R_{T-1}^{-\gamma} R_{r,T-1}\right] = R_f E_{T-1}\left[R_{T-1}^{-\gamma}\right]}$$

ω_{T-1}^{*} must satisfy this condition

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$R_t = h(\omega_t)$

$R_t^* = h(\omega_t^*)$

Score.....................

$\frac{a_1 W_{T-1}}{1+a_1}$

Question 1.2 (10 marks) Solve for the form of J(W_{T-1},T-1).

$$J(W_{T-1}, T-1) = \delta^{T-1} \frac{C_{T-1}^{*\,1-\gamma}}{1-\gamma} + \delta^T E_{T-1}\left[\frac{(R_{T-1}^*(W_{T-1} - C_{T-1}^*))^{1-\gamma}}{1-\gamma}\right]$$

$$= \frac{\delta^{T-1}\left(\frac{a_1}{1+a_1}\right)^{1-\gamma} W_{T-1}^{1-\gamma}}{1-\gamma} + \delta^T E_{T-1}\left[\frac{R_{T-1}^{*\,1-\gamma} W_{T-1}^{1-\gamma}}{(1-\gamma)(1+a_1)^{1-\gamma}}\right]$$

$$= \frac{\delta^{T-1} W_{T-1}^{1-\gamma}}{(1-\gamma)(1+a_1)^{1-\gamma}}\left(a_1^{1-\gamma} + \delta E_{T-1}\left[R_{T-1}^{*\,1-\gamma}\right]\right)$$

$$= \frac{\delta^{T-1} b_1 W_{T-1}^{1-\gamma}}{(1-\gamma)}, \text{ where } b_1 = \frac{a_1^{1-\gamma} + \delta E_{T-1}[R_{T-1}^{*\,1-\gamma}]}{(1+a_1)^{1-\gamma}}$$

$\delta E_{T-1}[R_{T-1}^{*\,1-\gamma}] = a_1^{-\gamma}$

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$$= \frac{a_1^{1-\gamma} + a_1^{-\gamma}}{(1+a_1)^{1-\gamma}}$$

$$= \frac{a_1^{-\gamma}(a_1+1)}{(1+a_1)^{1-\gamma}}$$

$$= \frac{a^{-\gamma}}{(1+a_1)^{-\gamma}} = \left(\frac{a_1}{1+a_1}\right)^{-\gamma}$$

Score.....................

Question 1.3 (10 marks) Derive the first-order condition for the optimal consumption level and portfolio weight at date T-2, C^*_{T-2} and ω^*_{T-2}., and give an explicit expression for C^*_{T-2}

At T-2, $J(W_{T-2}, T-2) = \max_{C_{T-2}, \omega_{T-2}} \left(\delta^{T-2} \frac{C_{T-2}^{1-\gamma}}{1-\gamma} + E_{T-2}\left[\underbrace{J(W_{T-1}, T-1)}_{\delta^{T-1} b_1 \frac{W_{T-1}^{1-\gamma}}{1-\gamma}}\right] \right)$

<u>Optimality conditions at T-2</u>

$$U_c(C^*_{T-2}) = E_{T-2}[J_W(W_{T-1}) R_{T-2}] \quad \text{— (3)}$$

$$\delta^{T-2} C_{T-2}^{-\gamma} = \delta^{T-1} E_{T-2}\left[b_1 W_{T-1}^{-\gamma} R_{T-2}\right]$$

$$C_{T-2}^{-\gamma} = \delta E_{T-2}\left[b_1 (W_{T-2} - C_{T-2})^{-\gamma} R_{T-2}^{-\gamma} R_{T-2}\right]$$

$$C_{T-2}^{-\gamma} = \delta b_1 (W_{T-2} - C_{T-2})^{-\gamma} E_{T-2}\left[R_{T-2}^{1-\gamma}\right]$$

$$C_{T-2} = \left(\delta b_1 E_{T-2}[R_{T-2}^{1-\gamma}]\right)^{\frac{1}{-\gamma}} (W_{T-2} - C_{T-2})$$

$$C^*_{T-2} = \frac{\left(\delta b_1 E_{T-2}[R_{T-2}^{1-\gamma}]\right)^{\frac{1}{-\gamma}} W_{T-2}}{1 + \left(\delta b_1 E_{T-2}[R_{T-2}^{1-\gamma}]\right)^{\frac{1}{-\gamma}}}$$

$$C^*_{T-2} = \frac{a_2}{1+a_2} W_{T-2} = c_2 W_{T-2}$$

where $$a_2 = \left(\delta b_1 E_{T-2}\left[R_{T-2}^{1-\gamma}\right]\right)^{\frac{1}{-\gamma}} = \left(\delta b_1 E\left[R_{T-2}^{1-\gamma}\right]\right)^{\frac{1}{-\gamma}}$$

$$= b_1^{\frac{1}{-\gamma}} \underbrace{\left(\delta E\left[R_{T-2}^{1-\gamma}\right]\right)^{\frac{1}{-\gamma}}}_{a_1} = \left(\frac{a_1}{1+a_1}\right) a_1 = c_1 a_1 ,$$

$$c_2 = \frac{a_2}{1+a_2}$$

$$E_{T-2}\left[R_{r,T-2} J_w(W_{T-1})\right] = R_f E_{T-2}\left[J_w(W_{T-1})\right] \quad \text{—— (4)}$$

$$E_{T-2}\left[R_{r,T-2} \cancel{\delta^{T-1}} b_1 W_{T-1}^{-\gamma}\right] = R_f E_{T-2}\left[\cancel{\delta^{T-1}} b_1 W_{T-1}^{-\gamma}\right]$$

$$E_{T-2}\left[R_{r,T-2} \cancel{b_1} \cancel{S_{T-2}^{-\gamma}} R_{T-2}^{-\gamma}\right] = R_f E_{T-2}\left[\cancel{b_1} \cancel{S_{T-2}^{-\gamma}} R_{T-2}^{-\gamma}\right]$$

$$E_{T-2}\left[R_{r,T-2} R_{T-2}^{-\gamma}\right] = R_f E_{T-2}\left[R_{T-2}^{-\gamma}\right]$$

$$\boxed{E\left[R_{r,T-2} R_{T-2}^{-\gamma}\right] = R_f E\left[R_{T-2}^{-\gamma}\right]}$$

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ω^*_{T-2} must satisfy this condition, and it is the same condition (2) as to choose ω^*_{T-1}.

Score.....................

Question 1.4 (10 marks) Solve for the form of J(W_{T-2},T-2).Based on the pattern for T-1 and T-2, provide expressions for the optimal consumption and portfolio weight at any date T-t, t=1,2,3,....

$$J(W_{T-2}, T-2) = \delta^{T-2}\frac{C^{*\,1-\gamma}_{T-2}}{1-\gamma} + E_{T-2}\left[\delta^{T-1}\frac{b_1 W_{T-1}^{1-\gamma}}{1-\gamma}\right]$$

$$= \delta^{T-2}\frac{C^{*\,1-\gamma}_{T-2}}{1-\gamma} + E_{T-2}\left[\frac{\delta^{T-1} b_1 (W_{T-2} - \underbrace{C^{*}_{T-2}}_{\frac{a_2}{1+a_2}W_{T-2}})^{1-\gamma} R^{*\,1-\gamma}_{T-2}}{1-\gamma}\right]$$

$$= \frac{\delta^{T-2} a_2^{1-\gamma} W_{T-2}^{1-\gamma}}{(1-\gamma)(1+a_2)^{1-\gamma}} + \delta^{T-1} E_{T-2}\left[\frac{b_1 R^{*\,1-\gamma}_{T-2} W_{T-2}^{1-\gamma}}{(1-\gamma)(1+a_2)^{1-\gamma}}\right]$$

$$= \frac{\delta^{T-2} W_{T-2}^{1-\gamma}}{(1-\gamma)(1+a_2)^{1-\gamma}}\Big(a_2^{1-\gamma} + \underbrace{\delta E_{T-2}\left[b_1 R^{*\,1-\gamma}_{T-2}\right]}_{a_2^{-\gamma}}\Big)$$

$$= \frac{\delta^{T-2} b_2 W_{T-2}^{1-\gamma}}{1-\gamma}, \quad b_2 = \frac{a_2^{1-\gamma} + \delta E_{T-2}\left[b_1 R^{*\,1-\gamma}_{T-2}\right]}{(1+a_2)^{1-\gamma}} = \left(\frac{a_2}{1+a_2}\right)^{-\gamma} \quad \#$$

We can see the pattern that

$$C^*_{T-1} = \frac{a_1}{1+a_1} W_{T-1}$$

$$C^*_{T-2} = \frac{c_1 a_1}{1+c_1 a_1} W_{T-2}$$

So, in general, $C^*_{T-t} = c_t W_{T-t}$

where $c_t = \dfrac{a_1 c_{t-1}}{1 + a_1 c_{t-1}}$ #

and ω^*_t is the same each period that satisfies (4). #