

Macroeconomics

Lecture 4

Example

$$\max \sum_{t=0}^{\infty} \beta^t U(c_t), \quad 0 < \beta < 1$$

Subject to

$$A_{t+1} = R_t (A_t + y_t - c_t),$$

$$\text{or, } c_t + \sum_{j=1}^{\infty} \left(\prod_{k=0}^{j-1} R_{t+k}^{-1} \right) c_{t+j} = y_t + \sum_{j=1}^{\infty} \left(\prod_{k=0}^{j-1} R_{t+k}^{-1} \right) y_{t+j} + A_t. \quad (10)$$

A_0 given,

- State variables are A_t , y_t , R_{t-1} .
- Control variables is savings, $s_t = A_t + y_t - c_t$
- Transition function $A_{t+1} = R_t \cdot (A_t + y_t - c_t)$ does not involve state variable. (in this case, we have $x_{t+1} = g(c_t)$, instead of $x_{t+1} = g(x_t, c_t)$)

Saving under certainty

- Bellman's equation

$$V(A_t, y_t, R_{t-1}) = \max_{s_t} [U(A_t + y_t - s_t) + \beta V(R_t s_t, y_{t+1}, R_t)]$$

where $s_t = R_t^{-1} A_{t+1}$,

- Note that $c_t = A_t + y_t - s_t$, or $s_t = A_t + y_t - c_t$. Hence by recalling (9), which is the same marginal condition as (1.4b), and suppose that the optimal choice $s_t = h(A_t, y_t, R_{t-1})$ then we have

$$\begin{aligned} \frac{\partial V(A_t, y_t, R_{t-1})}{\partial A_t} = & \frac{\partial U(c_t)}{\partial c_t} \frac{\partial (A_t + y_t - h(A_t, y_t, R_{t-1}))}{\partial A_t} \\ & + \beta \frac{\partial V(R_t h(A_t, y_t, R_{t-1}), y_{t+1}, R_t)}{\partial R_t h(A_t, y_t, R_{t-1})} \cdot R_t \cdot \frac{\partial h(A_t, y_t, R_{t-1})}{\partial A_t}, \end{aligned}$$

Saving under certainty

$$\begin{aligned}
 \frac{\partial V(A_t, y_t, R_{t-1})}{\partial A_t} &= \frac{\partial U(c_t)}{\partial c_t} \left[1 - \frac{\partial h(A_t, y_t, R_{t-1})}{\partial A_t} \right] \\
 &\quad + \beta \frac{\partial V(R_t h(A_t, y_t, R_{t-1}), y_{t+1}, R_t)}{\partial R_t h(A_t, y_t, R_{t-1})} R_t \frac{\partial h(A_t, y_t, R_{t-1})}{\partial A_t}, \\
 &= U'(c_t) - \frac{\partial h(A_t, y_t, R_{t-1})}{\partial A_t} [U'(c_t) - \beta V'(A_{t+1}, y_{t+1}, R_t) R_t], \\
 &= U'(c_t), \quad (\because \text{First-order condition w.r.t. } u_t)
 \end{aligned}$$

The above result also valid for $t+1$, hence,

$$\frac{\partial V(A_{t+1}, y_{t+1}, R_t)}{\partial A_{t+1}} = \frac{\partial U(c_{t+1})}{\partial c_{t+1}} = U'(c_{t+1}), \tag{11}$$

Next, by differentiating Bellman equation w.r.t. s_t , one has

$$\frac{\partial U(A_t + y_t - s_t)}{\partial (A_t + y_t - s_t)} \frac{\partial (A_t + y_t - s_t)}{\partial s_t} + \beta \frac{\partial V(s_t R_t, y_{t+1}, R_t)}{\partial s_t R_t} \frac{\partial s_t R_t}{\partial s_t} = 0,$$

$$-U'(c_t) + \beta V'(A_{t+1}, y_{t+1}, R_t) R_t = 0,$$

$$-U'(c_t) + \beta R_t U'(c_{t+1}) = 0, \quad [\because \text{Eq. 11}] \quad (12)$$

Eq.12 is the Euler equation for s_t ,

If $U(c_t) = \ln c_t$, then from (12)

$$c_{t+j} = \beta^j [R_t \cdot R_{t+1} \dots R_{t+j-1}] c_t$$

Substituting this into the left hand side of (10), which is

$$c_t + \sum_{j=1}^{\infty} \left(\prod_{k=0}^{j-1} R_{t+k}^{-1} \right) c_{t+j}, \text{ then}$$

$$\begin{aligned} c_t + \sum_{j=1}^{\infty} \left(\prod_{k=0}^{j-1} R_{t+k}^{-1} \right) c_{t+j} &= c_t + \sum_{j=1}^{\infty} \left[\left(\prod_{k=0}^{j-1} R_{t+k}^{-1} \right) \left(\beta^j [R_t \cdot R_{t+1} \dots R_{t+j-1}] c_t \right) \right] \\ &= c_t \left[1 + \sum_{j=1}^{\infty} \left[\left(\prod_{k=0}^{j-1} R_{t+k}^{-1} \right) \left(\beta^j [R_t \cdot R_{t+1} \dots R_{t+j-1}] \right) \right] \right] \\ &= c_t \left[1 + \{ \beta + \beta^2 + \dots \} \right] \\ &= c_t (1 - \beta)^{-1} \end{aligned}$$

Hence, (10) and (12) imply that

$$c_t = (1 - \beta) \left[y_t + \sum_{j=1}^{\infty} \left(\prod_{k=0}^{j-1} R_{t+k}^{-1} \right) y_{t+j} + A_t \right] \quad (13)$$

Optimal Growth under certainty

$$\max \sum_{t=0}^{\infty} \beta^t U(c_t), \quad 0 < \beta < 1$$

$$s.t. \quad c_t + k_{t+1} = f(k_t), \quad k_0 > 0 \text{ given}, \quad c_t \geq 0$$

$$U'(0) = +\infty, \quad U' > 0, \quad U'' < 0,$$

$$f'(0) = \infty, \quad f'(\infty) = 0, \quad f' > 0, \quad f'' < 0.$$

$$(\therefore \text{one may write } k_{t+1} = f(k_t) - c_t = g(c_t, k_t))$$

Optimal Growth under certainty

Bellman's equation,

$$V(k_t) = \max_{k_{t+1}} \{U[f(k_t) - k_{t+1}] + \beta V(k_{t+1})\}$$

Euler equation for k_{t+1} ,

$$\frac{\partial U[f(k_t) - k_{t+1}]}{\partial [f(k_t) - k_{t+1}]} \frac{d[f(k_t) - k_{t+1}]}{d[k_{t+1}]} + \beta \frac{\partial V(k_{t+1})}{\partial k_{t+1}} = 0,$$

$$\frac{\partial U(c_t)}{\partial c_t} (-1) + \beta \frac{\partial V(k_{t+1})}{\partial k_{t+1}} = 0,$$

$$\frac{\partial U(c_t)}{\partial c_t} = \beta \frac{\partial V(k_{t+1})}{\partial k_{t+1}}, \quad (14)$$

Optimal Growth

From (9), (The marginal condition), and the optimal choice $k_{t+1} = h(k_t)$

$$\begin{aligned}
 V'(k_t) &= \frac{\partial U[f(k_t) - h(k_t)]}{\partial [f(k_t) - h(k_t)]} \left[\frac{df(k_t)}{dk_t} - \frac{dh(k_t)}{dk_t} \right] + \beta \frac{\partial V(h(k_t))}{\partial h(k_t)} \frac{dh(k_t)}{dk_t} \\
 &= \frac{\partial U(c_t)}{\partial c_t} [f'(k_t)] - \frac{\partial h(k_t)}{k_t} \left[\frac{\partial U(c_t)}{\partial c_t} - \beta V'(k_{t+1}) \right] \\
 &= \frac{\partial U(c_t)}{\partial c_t} [f'(k_t)], \quad (\because \text{eq.(14)})
 \end{aligned}$$

The above result is also valid at $t+1$, hence

$$V'(k_{t+1}) = \frac{\partial U(c_{t+1})}{\partial c_{t+1}} [f'(k_{t+1})], \quad (15)$$

Hence, substitute (15) into (14), one has

$$-U'(c_t) + \beta U'(c_{t+1}) f'(k_{t+1}) = 0 \quad (16)$$

Optimal Growth

From (16),
$$f'(k_{t+1}) = \frac{U'(c_t)}{\beta U'(c_{t+1})},$$
$$= \frac{U'(f(k_t) - k_{t+1})}{\beta U'(f(k_{t+1}) - k_{t+2})}.$$

$\therefore$ Optimal policy function $k_{t+2} = h(k_{t+1})$ must be a non decreasing function of k_{t+1} . This is because

$$f''(k_{t+1}) =$$

$$\frac{[\beta U'(c_{t+1})][U''(c_t) \cdot (-1)] - [U'(c_t)] \left[\beta U''(c_{t+1}) \cdot \left(f'(k_{t+1}) - \frac{\partial h(k_{t+1})}{\partial k_{t+1}} \right) \right]}{[\beta U'(c_{t+1})]^2} \leq 0$$

$\Rightarrow$ Suppose that $\frac{\partial h(k_{t+1})}{\partial k_{t+1}} < 0$, then the RHS of the above equation will be

greater than zero, which is contradicted to $f''(k_{t+1}) \leq 0$. Hence, it must be

that
$$\frac{\partial h(k_{t+1})}{\partial k_{t+1}} \geq 0.$$

Appendix

$$A_{t+1} = R_t (A_t + y_t - c_t) \quad (a)$$

$$A_{t+2} = R_{t+1} (A_{t+1} + y_{t+1} - c_{t+1}) \quad (b)$$

$$A_{t+3} = R_{t+2} (A_{t+2} + y_{t+2} - c_{t+2}) \quad (c)$$

Put (a) into (b)

$$A_{t+2} = R_{t+1} (R_t [A_t + y_t - c_t] + y_{t+1} - c_{t+1}) \quad (d)$$

Put (d) into (c)

$$A_{t+3} = R_{t+2} (R_{t+1} (R_t [A_t + y_t - c_t] + y_{t+1} - c_{t+1}) + y_{t+2} - c_{t+2})$$

$$A_{t+3} = R_{t+2} R_{t+1} R_t [A_t + y_t - c_t] + R_{t+2} R_{t+1} [y_{t+1} - c_{t+1}] + R_{t+2} [y_{t+2} - c_{t+2}]$$

$$A_{t+3} = \left[\prod_{k=0}^2 R_{t+k} \right] [A_t + y_t - c_t] + \left[\prod_{k=1}^2 R_{t+k} \right] [y_{t+1} - c_{t+1}] + R_{t+2} [y_{t+2} - c_{t+2}]$$

$$A_{t+j} = \left[\prod_{k=0}^{j-1} R_{t+k} \right] [A_t + y_t - c_t] + \left[\prod_{k=1}^{j-1} R_{t+k} \right] [y_{t+1} - c_{t+1}] + \left[\prod_{k=2}^{j-1} R_{t+k} \right] [y_{t+2} - c_{t+2}]$$

$$+ \dots + \left[\prod_{k=j-2}^{j-1} R_{t+k} \right] [y_{t+j-2} - c_{t+j-2}] + R_{t+j-1} [y_{t+j-1} - c_{t+j-1}] \quad (e)$$

Let $j \rightarrow \infty$, $\lim_{j \rightarrow \infty} A_{t+j} = 0$, and eq.(e) becomes

$$0 = \left[\prod_{k=0}^{\infty} R_{t+k} \right] [A_t + y_t - c_t] + \left[\prod_{k=1}^{\infty} R_{t+k} \right] [y_{t+1} - c_{t+1}] \\ + \dots + \left[\prod_{k=2}^{\infty} R_{t+k} \right] [y_{t+j-2} - c_{t+j-2}] + R_{t+j-1} [y_{t+j-1} - c_{t+j-1}], \quad \dots (f)$$

Or,

$$\left[\prod_{k=0}^{\infty} R_{t+k} \right] c_t + \left[\prod_{k=1}^{\infty} R_{t+k} \right] c_{t+1} + \left[\prod_{k=2}^{\infty} R_{t+k} \right] c_{t+2} + \dots + \left[\prod_{k=j-1}^{\infty} R_{t+k} \right] c_{t+j-1} + \dots \\ = \left[\prod_{k=0}^{\infty} R_{t+k} \right] y_t + \left[\prod_{k=1}^{\infty} R_{t+k} \right] y_{t+1} + \dots + \left[\prod_{k=j-1}^{\infty} R_{t+k} \right] y_{t+j-1} + \dots + \left[\prod_{k=0}^{\infty} R_{t+k} \right] A_t, \quad \dots (g)$$

Then, multiplying both sides of eq.(g) by $\left[\prod_{k=0}^{\infty} R_{t+k} \right]^{-1}$,

$$c_t + (R_t)^{-1} c_{t+1} + (R_t \cdot R_{t+1})^{-1} c_{t+2} + \dots + (R_t \cdot R_{t+1} \dots R_{t+j-1})^{-1} c_{t+j} + \dots \\ = y_t + (R_t)^{-1} y_{t+1} + (R_t \cdot R_{t+1})^{-1} y_{t+2} + \dots + (R_t \cdot R_{t+1} \dots R_{t+j-1})^{-1} y_{t+j} + \dots + A_t,$$

Hence,

$$c_t + \sum_{j=1}^{\infty} \left(\prod_{k=0}^{j-1} R_{t+k}^{-1} \right) c_{t+j} = y_t + \sum_{j=1}^{\infty} \left(\prod_{k=0}^{j-1} R_{t+k}^{-1} \right) y_{t+j} + A_t, \quad (1.10) \quad Q.E.D.$$