

# Fundamentals of Mathematical Proofs: II

## TU152: Fundamental Mathematics

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## Method of Generalizing from the Generic Particular

In order to use the *method of generalizing from the generic particular* for

proving

$$\forall x \in D, P(x)$$

or disproving

$$\exists x \in D, Q(x),$$

it is helpful to use the following three steps:

- 1 Restate the claim in a **formal** way.
- 2 Specify the **starting point**.
- 3 Identify **the conclusion to be shown**.

# Method of Generalizing from the Generic Particular

**Example:** (Disproving an Existential Statement)

Disprove the following statement:

There is a positive integer  $n$  such that  $n^2 + 3n + 2$  is prime.

# Rational Numbers

## Definition

A real number  $r$  is **rational** if, and only if, it can be expressed as a quotient of two integers with a nonzero denominator. A real number that is not rational is **irrational**. More formally, if  $r$  is a real number, then

$$r \text{ is rational} \Leftrightarrow \exists \text{ integers } a \text{ and } b \text{ such that } r = \frac{a}{b}, b \neq 0.$$

Note: The word *rational* contains the word *ratio*, which is another word for *quotient*. A *rational number* can be written as a *ratio* of integers.

**Example:** Determine whether the following numbers are rational or irrational.

- 1 0
- 2  $10/3$
- 3  $-\frac{3}{47}$
- 4 0.1234
- 5  $0.12121212\dots$  (where the digits 12 are assumed to repeat forever)

# Rational Numbers

**Example:** Prove the following theorem.

Theorem

*The sum of any two rational numbers is rational.*

# Rational Numbers

A **corollary** is a statement whose truth can be immediately deduced from a theorem that has already been proved.

**Example:**

Corollary

The double of a rational number is rational.

## More Methods of Proof

### Vacuous Proof

A **vacuous proof** is a proof of an implication  $p \rightarrow q$  in which it is shown that  $p$  is false.

**Example** Use the method of vacuous proof to show that if  $x \in \emptyset$ , then David is playing soccer.

Answer:

### Trivial Proof

A **trivial proof** of an implication  $p \rightarrow q$  is one in which  $q$  is shown to be true without any reference to  $p$ .

**Example** Use the method of trivial proof to show that if  $n$  is an even integer then  $n$  is divisible by 1.

Answer:

# More Methods of Proof: Method of proof by cases

## Method of Proof by Cases

The method of proof by cases is a direct method of proving the conditional statement

$$(p_1 \vee p_2 \vee \cdots \vee p_n) \rightarrow q.$$

The method consists of proving the conditional statements

$$p_1 \rightarrow q, p_2 \rightarrow q, \dots, p_n \rightarrow q.$$

To prove a statement of the form

“If  $A_1$  or  $A_2$  or  $\dots$  or  $A_n$ , then  $C$ ”

we have to prove all of the following:

- If  $A_1$ , then  $C$ ,
- If  $A_2$ , then  $C$ ,
- $\vdots$
- $\vdots$
- If  $A_n$ , then  $C$ .

This process shows that  $C$  is true regardless of which of  $A_1, A_2, \dots, A_n$  happens to be the case.

# Method of Proof by Cases

**Example:** Proof the following statement.

If  $n$  is a positive integer, then  $n^3 + n$  is even.

## Method of Proof by Cases: Absolute value example

## Definition (Absolute Value)

For any real number  $x$ , the absolute value of  $x$ , denoted  $|x|$ , is defined as follows:

$$|x| = \begin{cases} -x, & x < 0 \\ x, & x \geq 0 \end{cases}$$

**Example:** Use the proof by cases to prove the triangle inequality:

For all real numbers  $x$  and  $y$ ,

$$|x + y| \leq |x| + |y|.$$

Floor  $\lfloor x \rfloor$  and Ceiling  $\lceil x \rceil$ 

## Definition (Floor)

Given any real number  $x$ , the floor of  $x$ , denoted  $\lfloor x \rfloor$ , is defined as:

$$\lfloor x \rfloor = n, \quad n \text{ is an integer such that } n \leq x < n + 1.$$

Symbolically, if  $x$  is a real number,

$$\lfloor x \rfloor = n \Leftrightarrow n \in \mathbb{Z}, \quad n \leq x < n + 1.$$

## Definition (Ceiling)

Given any real number  $x$ , the ceiling of  $x$ , denoted  $\lceil x \rceil$ , is defined as:

$$\lceil x \rceil = n, \quad n \text{ is an integer such that } n - 1 < x \leq n.$$

Symbolically, if  $x$  is a real number,

$$\lceil x \rceil = n \Leftrightarrow n \in \mathbb{Z}, \quad n - 1 < x \leq n.$$

Floor  $\lfloor x \rfloor$  and Ceiling  $\lceil x \rceil$ 

**Example:** Compute  $\lfloor x \rfloor$  and  $\lceil x \rceil$  of the following values of  $x$ .

1  $x = 37.999$

2  $x = 11$

3  $x = 0.6$

4  $x = -\frac{57}{2}$

5  $x = -14.001$

Floor  $\lfloor x \rfloor$  and Ceiling  $\lceil x \rceil$ 

**Example:** Use the proof by a counterexample to show that the statement

$$" \forall x, y \in \mathbb{R}, \lfloor x + y \rfloor = \lfloor x \rfloor + \lfloor y \rfloor "$$

is **false**.

Answer:

Floor  $\lfloor x \rfloor$  and Ceiling  $\lceil x \rceil$ 

**Example:** Use the method of proof by cases to prove the following statement.

Let  $n$  be an integer. Then

$$\left\lfloor \frac{n}{2} \right\rfloor = \begin{cases} \frac{n}{2}, & \text{if } n \text{ is even} \\ \frac{n-1}{2}, & \text{if } n \text{ is odd} \end{cases} .$$

