

Solution: Assignment 4

1. Let a , b and c be real numbers with $a < c$ and $|c| < |a| < |b|$.

Determine whether each of the following inequalities is true or not. Explain your answer.

(a) $\frac{a}{c^2-b^2} < \frac{c}{c^2-a^2}$

(b) $a > \frac{b^2-c^2}{a-c} - c$

Solution: From $|c| < |a| < |b|$, we have $c^2 < a^2 < b^2$ and by subtracting c^2 ,

$$\Rightarrow 0 < a^2 - c^2 < b^2 - c^2$$

- (a) False. Since $b^2 - c^2 > a^2 - c^2 > 0$, then $0 < \frac{1}{b^2-c^2} < \frac{1}{a^2-c^2}$ and by multiplying both sides by -1 we have ,

$$0 > \frac{-1}{b^2-c^2} > \frac{-1}{a^2-c^2} \quad \Rightarrow \quad 0 > \frac{1}{c^2-b^2} > \frac{1}{c^2-a^2}$$

so we can see that this inequality may not be always true. Consider a counterexample $a = -3, b = 4, c = 1$. Notice that $a < c$ gives $-3 < 1$, which is true and $|c| < |a| < |b|$ becomes $1 < 3 < 4$, which is true. However, $\frac{a}{c^2-b^2} = \frac{-3}{1-16} = \frac{1}{5}$, $\frac{c}{c^2-a^2} = \frac{1}{1-9} = -\frac{1}{8}$ and it is obviously $\frac{1}{5} \not< -\frac{1}{8}$. Hence this statement is not true. ■

- (b) True. From $0 < a^2 - c^2 < b^2 - c^2$ and $a^2 - c^2 = (a - c)(a + c)$, we have

$$0 < (a - c)(a + c) < b^2 - c^2$$

and since $a < c$ or $a - c < 0$, multiplying $a - c$ both sides will change the inequality from $<$ to $>$ and this gives

$$a + c > \frac{b^2 - c^2}{a - c} \quad \Rightarrow \quad a > \frac{b^2 - c^2}{a - c} - c.$$

Hence the inequality in (b) is true. ■

2. Find the solution set for each of following inequalities.

(a)

$$\frac{1}{x+3} < \frac{x}{x-3}$$

(b)

$$-\frac{1}{x} \leq 8x^2$$

(c)

$$\frac{|x+3|}{x} (3 - |x+3|) > \frac{2}{x}.$$

(d)

$$\frac{-x^2 + |x| - 3}{-x^4 + x^2 - x + 2 + |x^4 + 1|} \leq 0$$

(e)

$$\frac{2^x(2 - \sin(x))(x^3 - 1)}{x^5 - 4x^3 - x^2 + 4} \leq 0$$

Hint: $|\sin(x)| \leq 1$

(f) (Optional)

$$\frac{|2x + 5|}{3|x| - 1} \leq 1 - \frac{2}{1 - 3|x|}$$

Solution:

(a)

$$\frac{1}{x + 3} < \frac{x}{x - 3}$$

Solution:

$$\begin{aligned} \frac{1}{x + 3} &< \frac{x}{x - 3} \\ \frac{1}{x + 3} - \frac{x}{x - 3} &< 0 \\ \frac{x - 3 - x(x + 3)}{(x + 3)(x - 3)} &< 0 \\ \frac{-x^2 - 2x - 3}{(x + 3)(x - 3)} &< 0 \end{aligned}$$

For $-x^2 - 2x - 3$, $b^2 - 4ac = (-2)^2 - 4(-1)(-3) < 0$ which implies that $-x^2 - 2x - 3 < 0$ for any real number x and the inequality $\frac{-x^2 - 2x - 3}{(x + 3)(x - 3)} < 0$ has the same solution set as

$$\frac{1}{(x + 3)(x - 3)} > 0.$$

By setting $(x + 3)(x - 3) = 0$, we have $x = \pm 3$ to divide the set of real numbers into 3 subintervals.

	$x \in (-\infty, -3)$	$x \in (-3, 3)$	$x \in (3, \infty)$
$(x + 3)(x - 3)$	$(-)(-) = (+)$	$(+)(-) = (-)$	$(+)(+) = (+)$

Hence, from the table, the solution set is $(-\infty, -3) \cup (3, \infty)$. ■

(b)

$$-\frac{1}{x} \leq 8x^2$$

Solution:

$$-\frac{1}{x} \leq 8x^2 \Rightarrow 8x^2 + \frac{1}{x} \geq 0 \Rightarrow \frac{8x^3 + 1}{x} \geq 0$$

For the numerator $8x^3 + 1 = 0$, $x^3 = -1/8$ or $x = -1/2$. and the denominator is zero when $x = 0$. So we consider $x = -1/2$ and $x = 0$ to divide the set of real numbers into 3 subintervals.

	$x \in (-\infty, -1/2)$	$x \in (-1/2, 0)$	$x \in (0, \infty)$
$\frac{8x^3+1}{x}$	$\frac{(-)}{(-)} = (+)$	$\frac{(+)}{(-)} = (-)$	$\frac{(+)}{(+)} = (+)$

Hence, from the table and by the fact that the denominator $x \neq 0$ and using the inequality " \geq ", the solution set is $[-1/2, 0)$. ■

(c)

$$\frac{|x+3|}{x} (3 - |x+3|) > \frac{2}{x}.$$

Solution: The given inequality can be simplified as follow.

$$\begin{aligned} \frac{|x+3|}{x} (3 - |x+3|) &> \frac{2}{x} \\ \frac{3|x+3| - |x+3|^2}{x} - \frac{2}{x} &> 0 \\ \frac{|x+3|^2 - 3|x+3| + 2}{x} &< 0 \\ \frac{(|x+3| - 2)(|x+3| - 1)}{x} &< 0 \end{aligned}$$

From the definition: $|x+3| = \begin{cases} -(x+3), & x+3 < 0 \Leftrightarrow x < -3 \\ x+3, & x+3 \geq 0 \Leftrightarrow x \geq -3 \end{cases}$, we will now consider 2 cases.

(I) For $x < -3$, $|x+3| = -(x+3) = -x-3$, the inequality becomes

$$\frac{(-x-3-2)(-x-3-1)}{x} < 0 \quad \Rightarrow \quad \frac{(x+5)(x+4)}{x} < 0$$

and we consider $x = -5, -4, 0$ to subdivide the intervals.

	$x \in (-\infty, -5)$	$x \in (-5, -4)$	$x \in (-4, 0)$	$(0, \infty)$
$\frac{(x+5)(x+4)}{x}$	$\frac{(-)(-)}{(-)} = (-)$	$\frac{(+)(-)}{(-)} = (+)$	$\frac{(+)(+)}{(-)} = (-)$	$\frac{(+)(+)}{(+)} = (+)$

That is, the solution set in this case is $\{(-\infty, -5) \cup (-4, 0)\} \cap (-\infty, -3) = (-\infty, -5) \cup (-4, -3)$.

(II) For $x > -3$, $|x+3| = x+3$, the inequality becomes

$$\frac{(x+3-2)(x+3-1)}{x} < 0 \quad \Rightarrow \quad \frac{(x+1)(x+2)}{x} < 0$$

and we consider $x = -2, -1, 0$ to subdivide the intervals.

	$x \in (-\infty, -2)$	$x \in (-2, -1)$	$x \in (-1, 0)$	$(0, \infty)$
$\frac{(x+2)(x+1)}{x}$	$\frac{(-)(-)}{(-)} = (-)$	$\frac{(+)(-)}{(-)} = (+)$	$\frac{(+)(+)}{(-)} = (-)$	$\frac{(+)(+)}{(+)} = (+)$

That is, the solution set in this case is $\{(-\infty, -2) \cup (-1, 0)\} \cap (-3, \infty) = (-3, -2) \cup (-1, 0)$.

From (I) and (II), the solution set is

$$\{(-\infty, -5) \cup (-4, -3)\} \cup \{(-3, -2) \cup (-1, 0)\} = (-\infty, -5) \cup (-4, -2) \cup (-1, 0) \quad \blacksquare$$

(d)

$$\frac{-x^2 + |x| - 3}{-x^4 + x^2 - x + 2 + |x^4 + 1|} \leq 0$$

Solution:

- For the numerator $-x^2 + |x| - 3$, notice that $x^2 = |x|^2$ and consider $-|x|^2 + |x| - 3$ by checking that $b^2 - 4ac = 1 - 4(-1)(-3) < 0$ and $a = -1 < 0$. This implies that

(i) $-|x|^2 + |x| - 3 < 0$ for any $|x|$ or any x .

-For the denominator, since $x^4 \geq 0 \Rightarrow x^4 + 1 > 0$, then $|x^4 + 1| = x^4 + 1$ and

(ii) $-x^4 + x^2 - x + 2 + |x^4 + 1| = -x^4 + x^2 - x + 2 + x^4 + 1 = x^2 - x + 3 > 0$, which fol-

lows from the fact that $x^2 - x + 3 > 0$ for all $x \in \mathbb{R}$ since $b^2 - 4ac = 1 - 4(1)(3) < 0$ and $a = 1 > 0$. That is, from (i) and (ii),

$$\frac{-x^2 + |x| - 3}{-x^4 + x^2 - x + 2 + |x^4 + 1|} < 0 \leq 0,$$

is always true for any real number x and the solution set is the set of all real numbers \mathbb{R} .

■

(e)

$$\frac{2^x(2 - \sin(x))(x^3 - 1)}{x^5 - 4x^3 - x^2 + 4} \leq 0$$

Hint: $|\sin(x)| \leq 1$

Solution:

We first simplify the inequality by trying to factor the denominator first

$$x^5 - 4x^3 - x^2 + 4 = x^3(x^2 - 4) - 1(x^2 - 4) = (x^3 - 1)(x^2 - 4)$$

and

$$\frac{2^x(2 - \sin(x))(x^3 - 1)}{x^5 - 4x^3 - x^2 + 4} = \frac{2^x(2 - \sin(x))(x^3 - 1)}{(x^3 - 1)(x^2 - 4)} = \frac{2^x(2 - \sin(x))}{(x^2 - 4)} = \frac{2^x(2 - \sin(x))}{(x - 2)(x + 2)}$$

with $x^3 - 1 \neq 0 \Rightarrow x \neq 1$. So, the original inequality is reduced to

$$\frac{2^x(2 - \sin(x))}{(x - 2)(x + 2)} \leq 0 \quad \text{and } x \neq 1.$$

- For the numerator $2^x(2 - \sin(x))$, notice that $2^x > 0$ and consider $-2 - \sin(x)$ from $|\sin(x)| \leq 1$ or $-1 \leq \sin(x) \leq 1$,

$$1 \geq -\sin(x) \geq -1 \quad \Rightarrow \quad 2 - 1 \geq 2 - \sin(x) \geq 2 - (-1) \quad \Rightarrow \quad 1 \leq 2 - \sin(x) \leq 3$$

which implies $2 - \sin(x) > 0$ for all $x \in \mathbb{R}$. That is,

$$2^x(2 - \sin(x)) > 0 \quad \forall x \in \mathbb{R}$$

and we can divide both sides of the original inequality to obtain another equivalent problem

$$\frac{2^x(2 - \sin(x))(x^3 - 1)}{(x - 2)(x + 2)(x^3 - 1)} \leq 0 \quad \Leftrightarrow \quad \frac{1}{(x - 2)(x + 2)} \leq 0 \quad \text{and } x \neq 1.$$

	$x \in (-\infty, -2)$	$x \in (-2, 2)$	$x \in (2, \infty)$
$(x+2)(x-2)$	$(-)(-) = (+)$	$(+)(-) = (-)$	$(+)(+) = (+)$

Hence, from the table, the solution set is $(-2, 2) - \{1\} = (-2, 1) \cup (1, 2)$. ■

(f) (Optional)

$$\frac{|2x+5|}{3|x|-1} \leq 1 - \frac{2}{1-3|x|}$$

Solution: We will first simplify this in the form $\frac{|2x+5|-2}{3|x|-1} \leq 1$.

By using definition of absolute value, we have $|2x+5| = \begin{cases} -(2x+5), & 2x+5 < 0 \Leftrightarrow x < -5/2 \\ 2x+5, & 2x+5 \geq 0 \Leftrightarrow x \geq -5/2 \end{cases}$,

and $|x| = \begin{cases} -x, & x < 0 \\ x, & x \geq 0 \end{cases}$, we will consider 3 cases:

Case I: $x \in (-\infty, -5/2)$, Case II: $x \in [-5/2, 0)$, Case III: $x \in [0, \infty)$.

Case I: $x \in (-\infty, -5/2)$.

$$\begin{aligned} \frac{|2x+5|-2}{3|x|-1} &\leq 1 \\ \frac{-(2x+5)-2}{-3x-1} &\leq 1 \Leftrightarrow \frac{2x+7}{3x+1} \leq 1 \Leftrightarrow \frac{2x+7}{3x+1} - 1 \leq 0 \Leftrightarrow \frac{(2x+7)-(3x+1)}{3x+1} \leq 0 \\ \frac{-x+6}{3x+1} &\leq 0 \quad \text{or} \quad \frac{x-6}{3x+1} \geq 0 \end{aligned}$$

We consider the intervals divided by $x = -1/3, 6$ as follows.

	$x \in (-\infty, -1/3)$	$x \in (-1/3, 6)$	$x \in (6, \infty)$
$\frac{x-6}{3x+1}$	$\frac{(-)}{(-)} = (+)$	$\frac{(-)}{(+)} = (-)$	$\frac{(+)}{(+)} = (+)$

Note that we consider \geq , so $\{6\}$ has to be included in the solution set, but not the points that give zero denominator. Hence, from the table, $(-\infty, -1/3) \cup [6, \infty)$. That is, in this case, the solution set is $(-\infty, -5/2) \cap \{(-\infty, -1/3) \cup [6, \infty)\} = (-\infty, -5/2)$.

Case II: $x \in [-5/2, 0)$,

$$\begin{aligned} \frac{|2x+5|-2}{3|x|-1} &\leq 1 \\ \frac{(2x+5)-2}{-3x-1} &\leq 1 \Leftrightarrow \frac{2x+3}{3x+1} \geq -1 \Leftrightarrow \frac{2x+3}{3x+1} + 1 \geq 0 \Leftrightarrow \frac{(2x+3)+(3x+1)}{3x+1} \geq 0 \\ \frac{5x+4}{3x+1} &\geq 0 \end{aligned}$$

We consider the intervals divided by $x = -4/5, -1/3$ as follows.

	$x \in (-\infty, -4/5)$	$x \in (-4/5, -1/3)$	$x \in (-1/3, \infty)$
$\frac{5x+4}{3x+1}$	$\frac{(-)}{(-)} = (+)$	$\frac{(-)}{(+)} = (-)$	$\frac{(+)}{(+)} = (+)$

Note that we consider \leq , so $\{-4/5\}$ has to be included in the solution set, but not the points that give zero denominator. Hence, from the table, $(-1/3, 2]$. That is, in this case, the solution set is $[-5/2, 0) \cap \{(-\infty, -4/5] \cup (-1/3, \infty)\} = [-5/2, -4/5] \cup (-1/3, 0)$.

Case III: $x \in [0, \infty)$.

$$\begin{aligned} \frac{|2x+5|-2}{3|x|-1} &\leq 1 \\ \frac{(2x+5)-2}{3x-1} &\leq 1 \Leftrightarrow \frac{2x+3}{3x-1} \leq 1 \Leftrightarrow \frac{2x+3}{3x-1} - 1 \leq 0 \Leftrightarrow \frac{(2x+3)-(3x-1)}{3x-1} \leq 0 \\ \frac{-x+4}{3x-1} &\leq 0 \quad \text{or} \quad \frac{x-4}{3x-1} \geq 0 \end{aligned}$$

We consider the intervals divided by $x = 1/3, 4$ as follows.

	$x \in (-\infty, 1/3)$	$x \in (1/3, 4)$	$x \in (4, \infty)$
$\frac{x-4}{3x-1}$	$\frac{(-)}{(-)} = (+)$	$\frac{(-)}{(+)} = (-)$	$\frac{(+)}{(+)} = (+)$

Note that we consider \geq , so $\{4\}$ has to be included in the solution set, but not the points that give zero denominator. Hence, from the table, $(-\infty, 1/3) \cup [4, \infty)$. That is, in this case, the solution set is $[0, \infty) \cap \{(-\infty, 1/3) \cup [4, \infty)\} = [0, 1/3) \cup [4, \infty)$.

From cases (I)-(III), we have that the solution set is given by $(-\infty, -5/2) \cup \{[-5/2, -4/5] \cup (-1/3, 0)\} \cup \{[0, 1/3) \cup [4, \infty)\} = (-\infty, -4/5] \cup (-1/3, 1/3) \cup [4, \infty)$. ■

3. (Optional) Find the solution set for each of the following inequities.

(a) $\frac{x-1}{x+1} \leq 0$

Ans: $(-1, 1]$

(b) $\frac{-x^2-x}{2-x} \geq 0$

Ans: $[-1, 0] \cup (2, \infty)$

(c) $\frac{2x}{x-2} - \frac{3x}{x-4} \leq 1$

Ans: $(-\infty, 2) \cup (4, \infty)$

(d) $\frac{(2x^2+4x+7)(x-1)}{x^3-5x^2+x-5} \leq 0$

Ans: $[1, 5)$

(e) $\frac{x^2+8}{x^2+x+6} \geq 0$

Ans: $(-\infty, \infty)$

(f) $\frac{x^2-2x-4}{x^2-x-6} \geq 0$

Ans: $(-\infty, -2) \cup [1 - \sqrt{5}, 3) \cup (1 + \sqrt{5}, \infty)$

(g) $\frac{2x-x^2-3}{x-2x^2-1} \leq 1$

Ans: $(-\infty, -2] \cup [1, \infty)$

(h) $\frac{(-x^2+5x-7)(2x+1)}{x^4-1} \geq 0$

Ans: $(-\infty, -1) \cup [-1/2, 1)$

- (i) $\frac{x^2-1}{2-3x+x^2} > \frac{1}{x}$
 Ans: $(-\infty, 0) \cup (2, \infty)$
- (j) $\frac{x^4+6x^2}{1-2x} \leq x^2$
 Ans: $\{0\} \cup (1/2, \infty)$
- (k) $x^5 + 3x^4 - 23x^3 - 51x^2 + 94x + 120 \geq 0$
- (l) $\frac{\ln(x^2+x+4)}{|\sin(x)-\cos(x)|-3} < 0$
 Ans: \mathbb{R} (Hint: $x^2 + x + 4 > 1$)

4. (Optional) Find the solution set for each of the following inequities.

- (a) $|x - 1| < 7$
 Ans: $(-6, 8)$
- (b) $3 < |x + 2| < 4$
 Ans: $(-6, -5) \cup (1, 2)$
- (c) $|2x + 3| \leq 2$
 Ans: $[-5/2, -1/2]$
- (d) $|3x - 1| \geq 1$
 Ans: $(-\infty, 0] \cup [2/3, \infty)$
- (e) $\frac{|x+2|}{2} > |x|$
 Ans: $(-2/3, 2)$
- (f) $|x + 1| - |2x - 1| < 0$
 Ans: $(-\infty, 0) \cup (2, \infty)$
- (g) $|2 - x| - x \leq 0$
 Ans: $[1, \infty)$
- (h) $|x + 1| > 3 - |x|$
 Ans: $(-\infty, -2) \cup (1, \infty)$
- (i) $x^2 - 2 \geq \frac{1}{2}|x - 1|$
 Ans: $(-\infty, \frac{-1-\sqrt{41}}{4}) \cup (3/2, \infty)$
- (j) $|x + 3| \geq 2 - x$
 Ans: $[-1/2, \infty)$
- (k) $|2x^2 + x - 1| \geq 2$
 Ans: $(-\infty, -3/2] \cup [1, \infty)$
- (l) $\frac{|x|}{2} + 3 > x^2$
 Ans: $(-2, 2)$
- (m) $|x| > \frac{2}{|x+1|}$
 Ans: $(-\infty, -2) \cup (1, \infty)$
- (n) $\frac{x^2}{x+1} < |x|$
 Ans: $(-\infty, -1) \cup (-1/2, 0) \cup (0, \infty)$

(o) $\frac{4}{|x|} \leq \frac{1}{|x|}$

Ans: \emptyset

(p) $|x^2 + 1| < |x + 1|$

Ans: $(0, 1)$

(q) $(x^2 + 1)(|x + 2| - |x|) \geq 0$

Ans: $[-1, \infty)$

(r) $\frac{|1-x|}{x^3+2x^2+5x+4} \leq 0$

Ans: $(-\infty, -1) \cup \{1\}$ (Hint: for $f(x) = x^3 + 2x^2 + 5x + 4$, $f(-1) = 0$)