

## Relations & Functions II

### 1 Inverse of A Function and Inverse Function

**Definition 1.1** (inverse of function). Suppose  $f : X \rightarrow Y$  is a function. Then we can write

$$y = f(x), x \in X \quad \text{or} \quad f = \{(x, y) | x \in X, y = Y\}.$$

The **inverse of function**  $f$ , denoted  $f^{-1}$ , is defined as

$$\boxed{f^{-1}(y) = x \Leftrightarrow y = f(x)} \quad \text{or} \quad \boxed{f^{-1} = \{(y, x) | (x, y) \in f\}}.$$

Note that the inverse of function  $f$  may or may not be a function. If  $f^{-1}$  is a function  $f^{-1}$  is called the **inverse function** of  $f$ .

If  $f$  is a one-to-one correspondence from a set  $X$  to a set  $Y$ , then there is a function from  $Y$  to  $X$  that “undoes” the action of  $f$ . That is, it sends each element of  $Y$  back to the element of  $X$  that it came from. This function is called the inverse function for  $F$ .

**Theorem 1.1.** Suppose  $f : X \rightarrow Y$  is a one-to-one correspondence. I.e., suppose  $f$  is one-to-one and onto. Then there is a function  $f^{-1} : Y \rightarrow X$  that is defined as follows:

Given any element  $y \in Y$ ,

$f^{-1}(y) =$  the unique element  $x \in X$  such that  $f(x)$  equals  $y$ . I.e.,

$$\boxed{f^{-1}(y) = x \Leftrightarrow y = f(x)}.$$

The function  $f^{-1}$  is called the **inverse function** for  $f$ .

**Theorem 1.2.** If  $X$  and  $Y$  are sets and  $f : X \rightarrow Y$  is one-to-one and onto, then  $f^{-1} : Y \rightarrow X$  is also one-to-one and onto.

**Example 1.1.** Let  $X = \{1, 2, 3\}$  and  $Y = \{-1, -2, -3\}$ .

Let  $f = \{(1, -3), (2, -1), (3, -3)\}$  be a function from  $X$  to  $Y$ .

Find the inverse of  $f$ ,  $f^{-1}$  and determine whether it is the inverse function or not.

**Example 1.2.** Let  $f = \{(x, y) \in \mathbb{R} \times \mathbb{R} \mid y = 2x - 3\}$  be a function. Find the inverse of the function  $f$  and determine whether it is the inverse function of  $f$  or not.

**Example 1.3.** The function  $f : \mathbb{R} \rightarrow \mathbb{R}$  defined by the formula  $f(x) = 4x - 1$  for all real numbers  $x$  was shown to be one-to-one in the previous examples. Find its inverse function.

**Example 1.4.** A function  $F : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R} \times \mathbb{R}$  defined as, for all  $(x, y) \in \mathbb{R} \times \mathbb{R}$ ,

$$F(x, y) = (x + y, x - y)$$

is shown to be a bijective function in the previous example. Determine its inverse function  $F^{-1} : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R} \times \mathbb{R}$ .

## 2 Composite Function

**Definition 2.1** (Composite Function). Let  $f : X \rightarrow \hat{Y}$  and  $g : Y \rightarrow Z$  be functions with the property that the range of  $f$  is a subset of the domain of  $g$ ,  $\hat{Y} \subseteq Y$ . Define a new function  $g \circ f : X \rightarrow Z$  as follows:

$$(g \circ f)(x) = g(f(x)) \text{ for all } x \in X,$$

where

- $g \circ f$  is read “ $g$  circle  $f$ ” and
- $g(f(x))$  is read “ $g$  of  $f$  of  $x$ .”

The function  $g \circ f$  is called the **composition** of  $f$  and  $g$ .

**Remarks:** Composition of functions is **not a commutative operation**.

For general functions  $f$  and  $g$ ,  $f \circ g$  need not necessarily equal  $g \circ f$  (although the two may be equal).

**Definition 2.2.** For  $f : X \rightarrow \hat{Y}$  and  $g : Y \rightarrow Z$  with  $\hat{Y} \not\subseteq Y$ , but

$$R_f \cap D_g \neq \emptyset$$

, where  $R_f$  is the range of  $f$  and  $D_g$  is the domain of  $g$  ( $D_g = Y$ ). We can still define a new function  $g \circ f : \hat{X} \rightarrow Z$  as follows:

$$(g \circ f)(x) = g(f(x)) \text{ for all } x \in \hat{X} \subseteq X.$$

Notice that when  $\hat{Y} \subseteq Y$ , the domain  $\hat{X}$  of  $g \circ f$  will be the same as the domain  $X$  of  $f$ . However  $\hat{Y} \not\subseteq Y$ , but  $R_f \cap D_g \neq \emptyset$ ,  $\hat{X}$  may be smaller than  $X$ .

Constructing composite function

Notation Let  $f : X \rightarrow Y$  be a function.

- $D_f$  denotes the **domain** of the function  $f$ .
- $R_f$  denotes the **range** of the function  $f$ .

**Composite Function**

Let  $f$  and  $g$  be functions such that  $R_f \cap D_g \neq \emptyset$ . Then we can define a composite function

$$(g \circ f)(x) = g(f(x)), \quad x \in \widehat{X} \subseteq D_f.$$

- Domain of  $g \circ f : D_{g \circ f} \subseteq D_f$ .
- Range of  $g \circ f : R_{g \circ f} \subseteq R_g$ .
- If  $R_f \subseteq D_g$ , then  $D_{g \circ f} = D_f$ .

**Example 2.1.** Let  $f : \mathbb{Z} \rightarrow \mathbb{Z}$  be the successor function and let  $g : \mathbb{Z} \rightarrow \mathbb{Z}$  be the squaring function. Then  $f(n) = n + 1$  for all  $n \in \mathbb{Z}$  and  $g(n) = n^2$  for all  $n \in \mathbb{Z}$ .

- Find the compositions  $g \circ f$  and  $f \circ g$ .
- Is  $g \circ f = f \circ g$ ? Explain.

**Example 2.2.** Let  $X = \{1, 2, 3\}$ ,  $\widehat{Y} = \{a, b, c, d\}$ ,  $Y = \{a, b, c, d, e\}$ , and  $Z = \{x, y, z\}$ . Define functions  $f : X \rightarrow \widehat{Y}$  and  $g : Y \rightarrow Z$  by

- $f(1) = c, f(2) = b, f(3) = a,$
- $g(a) = y, g(b) = y, g(c) = z, g(d) = z, g(e) = z.$

Draw the arrow diagrams for  $f$ ,  $g$ , and  $g \circ f$ .

Determine the range for each of  $f$ ,  $g$ , and  $g \circ f$ .

**Example 2.3.** Let  $f(x) = \frac{2}{\sqrt{x-1}}$  and  $g(x) = x^2 + 3$ . Find  $g \circ f$  and  $f \circ g$ . Find  $(g \circ f)(2)$ .

**Example 2.4.** (Composition with the Identity Function):

Let  $X = \{a, b, c, d\}$  and  $Y = \{u, v, w\}$ , and suppose  $f : X \rightarrow Y$  is given by

$$f(a) = u, \quad f(b) = v, \quad f(c) = v, \quad f(d) = u.$$

Let  $I_X : X \rightarrow X$  and  $I_Y : Y \rightarrow Y$  with  $I_X(x) = x$  and  $I_Y(y) = y$  be identity functions. Find  $f \circ I_X$  and  $I_Y \circ f$ .

**Theorem 2.1.** Composition with an Identity Function

If  $f$  is a function from a set  $X$  to a set  $Y$ , and  $I_X$  is the identity function on  $X$ , and  $I_Y$  is the identity function on  $Y$ , i.e.,

$$I_X(x) = x \quad \forall x \in X, \quad I_Y(y) = y \quad \forall y \in Y$$

then

$$f \circ I_X = f \quad \text{and} \quad I_Y \circ f = f.$$

Let  $f$  be a function from a set  $X$  to a set  $Y$ , and suppose  $f$  has an inverse function  $f^{-1}$ . Recall that  $f^{-1}$  is the function from  $Y$  to  $X$  with the property that

$$f^{-1}(y) = x \quad \Leftrightarrow \quad f(x) = y.$$

What happens when  $f$  is composed with  $f^{-1}$ ? Or when  $f^{-1}$  is composed with  $f$ ?

**Example 2.5.** (Exercise) Let  $X = \{a, b, c\}$  and  $Y = \{x, y, z\}$ . Define  $f : X \rightarrow Y$  by

$$f(a) = z, \quad f(b) = x, \quad f(c) = y.$$

1. Draw the arrow diagram for  $f$ .
2. Show that the inverse function  $f^{-1}$  exists for the function  $f$ .
3. Find  $f^{-1}$  and draw the arrow diagram for  $f^{-1}$ .
4. Find and draw the arrow diagrams for  $f^{-1} \circ f$  and  $f \circ f^{-1}$ . Compare with  $I_X$  and  $I_Y$ .

**Theorem 2.2.** Composition of a Function with Its Inverse

If  $f : X \rightarrow Y$  is a one-to-one and onto function with inverse function  $f^{-1} : Y \rightarrow X$ , then

$$f^{-1} \circ f = I_X \quad \text{and} \quad f \circ f^{-1} = I_Y.$$

**Example 2.6.** Show that the following statement is true.

**Theorem 2.3.** The composition of two injective(one-to-one) functions is also injective If  $f : X \rightarrow Y$  and  $g : Y \rightarrow Z$  are both **one-to-one** functions, then  $g \circ f$  is **one-to-one**.

**Example 2.7.** Show that the following statement is true.

**Theorem 2.4.** The composition of two surjective(onto) functions is also surjective(onto) If  $f : X \rightarrow Y$  and  $g : Y \rightarrow Z$  are both **onto** functions, then  $g \circ f$  is **onto**.

### 3 Exercise

1. Let  $f(x) = x^2 + 5$  and  $g(x) = \begin{cases} 3 - x, & x < 2 \\ \frac{1}{x}, & 2 \leq x < 3. \end{cases}$  Find  $f \circ g$  and  $g \circ f$ .

2. Define a function  $g : [0, 3) \rightarrow \mathbb{R}$  and  $h : [0, 3) \rightarrow \mathbb{R}$  as

$$g(x) = \frac{1}{x+1}, \quad \text{and} \quad h = \frac{12}{x+1} - 5,$$

and define a function  $f : [-1, \infty) \rightarrow [0, \infty)$  be a function

$$f(x) = \begin{cases} x + 1, & x \in [-1, 4) \\ \sqrt{x} + 3, & x \in [4, \infty). \end{cases}$$

(a) Find domains, co-domains, and ranges of  $f$ ,  $g$  and  $h$ .

(b) Find the composite functions  $f \circ g$ ,  $g \circ f$  and  $f \circ h$  together with their domains and ranges.

3. Define a function  $f : [-1, \infty) \rightarrow [0, \infty)$  be a function  $f(x) = \begin{cases} x + 1, & x \in [-1, 4) \\ \sqrt{x} + 3, & x \in [4, \infty). \end{cases}$

Show that  $f$  is injective and find its inverse function  $f^{-1}$ .