Chapter 3: Linear Space (Vector Spaces): Part 4

1 Row Space, Column Space, and Null Space

Consider a matrix $\mathbf{A} \in \mathbb{R}^{m\times n}$:

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

The **row vectors** of $\mathbf{A}$ are:

$$\begin{aligned} \mathbf{r}_1 &= \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \end{bmatrix} \\ \mathbf{r}_2 &= \begin{bmatrix} a_{21} & a_{22} & \cdots & a_{2n} \end{bmatrix} \\ \vdots & \vdots \ \vdots \\ \mathbf{r}_m &= \begin{bmatrix} a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}. \end{aligned}$$

The **column vectors** of $\mathbf{A}$ are:

$$\mathbf{c}_1 = \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix}, \quad \mathbf{c}_2 = \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix}, \quad \cdots, \quad \mathbf{c}_n = \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix}.$$

Definition 1.1. Suppose $\mathbf{A}$ is an $m \times n$ matrix.

- The **row space** of $\mathbf{A}$, $\text{row}(\mathbf{A})$, is the subspace of $\mathbb{R}^n$ spanned by the row vectors of $\mathbf{A}$.
- The **column space** of $\mathbf{A}$, $\text{col}(\mathbf{A})$, is the subspace of $\mathbb{R}^m$ spanned by the column vectors of $\mathbf{A}$.
- The **null space** of $\mathbf{A}$, $\text{null}(\mathbf{A})$, is the solution space of the homogeneous system of equations $\mathbf{Ax} = \mathbf{0}$, which is a subspace of $\mathbb{R}^n$.

Goals:

- Find the relationships among the solutions of a linear system $\mathbf{Ax} = \mathbf{b}$ and the **row space**, **column space**, and **null space** of $\mathbf{A}$?
- Find the relationships among the **row space**, **column space**, and **null space** of a matrix $\mathbf{A}$.

Theorem 1.1. A system of linear equations $\mathbf{Ax} = \mathbf{b}$ is **consistent** if and only if $\mathbf{b}$ is in the column space of $\mathbf{A}$.

Proof:

Example 1.1. Consider $\mathbf{Ax} = \mathbf{b}$, where $\mathbf{A} = \begin{bmatrix} -1 & 3 & 2 \\ 1 & 2 & -3 \\ 2 & 1 & -2 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} 1 \\ -9 \\ 3 \end{bmatrix}$. Show that $\mathbf{b}$ is in the column space of $\mathbf{A}$ by expressing it as a linear combination of the column vectors of $\mathbf{A}$.

Theorem 1.2. Let $\mathbf{x}_0$ is any solution of a consistent linear system $\mathbf{Ax} = \mathbf{b}$.
If $S = \{\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_k\}$ is a basis for the null space of $\mathbf{A}$, then every solution of $\mathbf{Ax} = \mathbf{b}$ can be expressed in the form

$$\mathbf{x} = \mathbf{x}_0 + c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_k\mathbf{v}_k,$$

for any constant c_1, c_2, ..., c_k.

- The vector $\mathbf{x}_0$ is called a **particular solution** of $\mathbf{Ax} = \mathbf{b}$.
- The sum $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_k\mathbf{v}_k$ is called the **general solution** of $\mathbf{Ax} = \mathbf{0}$.

That is, we can summarize the above statement as follow.

The **general solution** of a consistent linear system can be expressed as the sum of a **particular solution** of that system and the **general solution** of the corresponding homogeneous system.

Example 1.2. Consider the following linear system and its corresponding homogeneous system:

$$\mathbf{Ax} = \mathbf{b}, \qquad \mathbf{Ax} = \mathbf{0},$$

where $\mathbf{A} = \begin{bmatrix} 1 & 3 & -2 & 0 & 2 & 0 \\ 2 & 6 & -5 & -2 & 4 & -3 \\ 0 & 0 & 5 & 10 & 0 & 15 \\ 2 & 6 & 0 & 8 & 4 & 18 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} 0 \\ -1 \\ 5 \\ 6 \end{bmatrix}$, and $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix}$. It can be shown that the solution of $\mathbf{Ax} = \mathbf{b}$ is $[0, 0, 0, 0, 0, 1/3]$ and the general solution of the homogeneous system is

$$x_1 = -3r - 4s - 2t, x_2 = r, \quad x_3 = -2s, \quad x_4 = s, \quad x_5 = t, \quad x_6 = 0.$$

Then we can also write the solution of $\mathbf{Ax} = \mathbf{b}$ as

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} =$$

<u>**Remarks:**</u>

- Elementary row operations do not change the null space of a matrix.
- Elementary row operations do not change the row space of a matrix.
- Elementary row operations may change the column space of a matrix.

Example 1.3. Find a basis for the null space of the matrix $\mathbf{A} = \begin{bmatrix} 1 & 3 & -2 & 0 & 2 & 0 \\ 2 & 6 & -5 & -2 & 4 & -3 \\ 0 & 0 & 5 & 10 & 0 & 15 \\ 2 & 6 & 0 & 8 & 4 & 18 \end{bmatrix}$.

The following theorem makes it possible to find bases for the row and column spaces of a matrix in row echelon form.

Theorem 1.3. If a matrix $\mathbf{R}$ is in row echelon form, then

- the row vectors with the leading 1's (the nonzero row vectors) form a basis for the row space of $\mathbf{R}$, and
- the column vectors with the leading 1's of the row vectors form a basis for the column space of $\mathbf{R}$.

Example 1.4. Find bases for the row and column spaces of the matrix $\mathbf{R} = \begin{bmatrix} 1 & -2 & 5 & 0 & 3 \\ 0 & 1 & 3 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

Remark: An elementary row operation can alter its column space. However, elementary row operations do not alter dependence relationships among the column vectors.

Theorem 1.4. If $\mathbf{A}$ and $\mathbf{B}$ are row equivalent matrices, then:

(a) A given set of column vectors of $\mathbf{A}$ is linearly independent if and only if the corresponding column vectors of $\mathbf{B}$ are linearly independent. I.e.

$$\{\mathbf{a}_{i_1}, \mathbf{a}_{i_2}, ..., \mathbf{a}_{i_r}\} \text{ is linear independent if and only if } \{\mathbf{b}_{i_1}, \mathbf{b}_{i_2}, ..., \mathbf{b}_{i_r}\} \text{ is linear independent.}$$

(b) A given set of column vectors of $\mathbf{A}$ forms a basis for the column space of $\mathbf{A}$ if and only if the corresponding column vectors of $\mathbf{B}$ form a basis for the column space of $\mathbf{B}$.

$$\{\mathbf{a}_{i_1}, \mathbf{a}_{i_2}, ..., \mathbf{a}_{i_r}\} \text{ is a basis of } \mathbf{A} \text{ if and only if } \{\mathbf{b}_{i_1}, \mathbf{b}_{i_2}, ..., \mathbf{b}_{i_r}\} \text{ is a basis of } \mathbf{B}.$$

Note: It is possible that $\text{col}(\mathbf{A}) \neq \text{col}(\mathbf{B})$ or $span\{\mathbf{a}_{i_1}, \mathbf{a}_{i_2}, ..., \mathbf{a}_{i_r}\} \neq span\{\mathbf{b}_{i_1}, \mathbf{b}_{i_2}, ..., \mathbf{b}_{i_r}\}$

Example 1.5. Find a basis for the row and column spaces of the matrix $\mathbf{A}$ if

$$\mathbf{A} = \begin{bmatrix} 1 & -3 & 4 & -2 & 5 & 4 \\ 2 & -6 & 9 & -1 & 8 & 2 \\ 2 & -6 & 9 & -1 & 9 & 7 \\ -1 & 3 & -4 & 2 & -5 & -4 \end{bmatrix} \text{ and the corresponding RREF of } \mathbf{A} \text{ is } \mathbf{R} = \begin{bmatrix} 1 & -3 & 4 & -2 & 5 & 4 \\ 0 & 0 & 1 & 3 & -2 & -6 \\ 0 & 0 & 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Remark:
Consider a matrix $\mathbf{A} \in \mathbb{R}^{m\times n}$ and its corresponding RREF $\mathbf{B}$.
Let $\hat{\mathbf{a}}_1, ..., \hat{\mathbf{a}}_m$ be the row vectors of $\mathbf{A}$.
Let $\hat{\mathbf{b}}_1, ..., \hat{\mathbf{b}}_m$ be the row vectors of $\mathbf{B}$.
Let $\mathbf{a}_1, ..., \mathbf{a}_n$ be the row vectors of $\mathbf{A}$.
Let $\mathbf{b}_1, ..., \mathbf{b}_n$ be the row vectors of $\mathbf{B}$.
Suppose $\mathbf{B}$ has leading 1's on the rows $i_1, ..., i_r$ and columns $j_1, ..., j_r$.

- $\text{row}(\mathbf{A}) = \text{row}(\mathbf{B})$
- $\text{row}(\mathbf{B}) = \text{span}(\{\hat{\mathbf{b}}_{i_1}, ..., \hat{\mathbf{b}}_{i_r}\})$

  $\{\hat{\mathbf{b}}_{i_1}, ..., \hat{\mathbf{b}}_{i_r}\}$ is a basis for $\text{row}(\mathbf{A})$ and $\text{row}(\mathbf{B})$.
- *It is possible that $\text{row}(\mathbf{A}) \neq \text{span}(\{\hat{\mathbf{a}}_{i_1}, ..., \hat{\mathbf{a}}_{i_r}\})$
- It is possible that $\text{col}(\mathbf{A}) \neq \text{col}(\mathbf{R})$
- $\text{col}(\mathbf{B}) = \text{span}(\{\mathbf{b}_{j_1}, ..., \mathbf{b}_{j_r}\})$

  $\{\mathbf{b}_{j_1}, ..., \mathbf{b}_{j_r}\}$ is a basis for $\text{col}(\mathbf{B})$.
- $\text{col}(\mathbf{A}) = \text{span}(\{\mathbf{a}_{j_1}, ..., \mathbf{a}_{j_r}\})$

  $\{\mathbf{a}_{j_1}, ..., \mathbf{a}_{j_r}\}$ is a basis for $\text{col}(\mathbf{A})$.

*To find a basis for the row space of a matrix $\mathbf{A}$, $\text{row}(\mathbf{A})$, that consists entirely of row vectors of that matrix $\mathbf{A}$, we will work on $\mathbf{A}^T$ and find the basis of $\mathbf{A}^T$ by using RREF of $\mathbf{A}^T$ as shown in the next example.

Example 1.6. Let $\mathbf{A} = \begin{bmatrix} 1 & -2 & 0 & 0 & 3 \\ 2 & -5 & -3 & -2 & 6 \\ 0 & 5 & 15 & 10 & 0 \\ 2 & 6 & 18 & 8 & 6 \end{bmatrix}$ and suppose the RREF of $\mathbf{A}^T$ is given by $\begin{bmatrix} 1 & 2 & 0 & 2 \\ 0 & 1 & -5 & -10 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$.
Find the rows of $\mathbf{A}$ that form a basis for $\text{row}(\mathbf{A})$.

Basis for the Space Spanned by a Set of Vectors
Given a set of vectors $S = \{\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_k\}$ in $\mathbb{R}^n$, we can find a subset of these vectors that forms a basis for span(S) by using the following steps.

Step 1. Form thematrix $\mathbf{A}$ whose columns are the vectors in the set $S = \{\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_k\}$.

Step 2. Reduce the matrix $\mathbf{A}$ to reduced row echelon form $\mathbf{R}$.

Step 3. Denote the column vectors of $\mathbf{R}$ by $\mathbf{w}_1, \mathbf{w}_2, ..., \mathbf{w}_k$.

Step 4. Identify the columns of $\mathbf{R}$ that contain the leading 1's, say $\{\mathbf{w}_{j_1}, \ldots, \mathbf{w}_{j_r}\}$. Then, the corresponding column vectors of $\mathbf{A}$ form a basis for span(S), i.e. $\{\mathbf{v}_{j_1}, \ldots, \mathbf{v}_{j_r}\}$.

Each vector $\mathbf{v}_i$ that is not in that basis can be expressed as a linear combination of the basis vectors as follows.
- For each column $\mathbf{w}_i$ of $\mathbf{R}$ where i is the column that does not have leading 1, write

$$\mathbf{w}_i = c_1\mathbf{w}_{j_1} + \cdots + c_r\mathbf{w}_{j_r}.$$

Note: $c_1, ..., c_r$ can be solved easily from the RREF matrix $\mathbf{R}$.
- Then $\mathbf{v}_i$ can be expressed in terms of the basis $\{\mathbf{v}_{j_1}, \ldots, \mathbf{v}_{j_r}\}$ by using the same coefficients $c_1, ..., c_r$, i.e.

$$\mathbf{v}_i = c_1\mathbf{v}_{j_1} + \cdots + c_r\mathbf{v}_{j_r}.$$

Example 1.7. Let $S = \{\mathbf{v}_1, \ldots, \mathbf{v}_5\}$ where
$\mathbf{v}_1 = (1, -2, 0, 3)$,
$\mathbf{v}_2 = (2, -5, -3, 6)$,
$\mathbf{v}_3 = (0, 1, 3, 0)$,
$\mathbf{v}_4 = (2, -1, 4, -7)$,
$\mathbf{v}_5 = (5, -8, 1, 2)$.

(a) Find a subset of S that forms a basis these vectors in S.
(b) Express each vector not in the basis as a linear combination of the basis vectors.

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2 Rank, Nullity, and the Fundamental Matrix Spaces

Theorem 2.1. The row space and the column space of a matrix $\mathbf{A}$ have the same dimension.

Proof: Note that the elementary row operations do not change the dimension of the row space or of the column space of a matrix. Thus, if $\mathbf{R}$ is any row echelon form of $\mathbf{A}$, it must be true that

$$\dim(\text{row space of } \mathbf{A}) = \dim(\text{row space of } \mathbf{R}) \; \dim(\text{column space of } \mathbf{A}) = \dim(\text{column space of } \mathbf{R}).$$

But the dimension of the row space of $\mathbf{R}$ is the number of nonzero rows, and the dimension of the column space of $\mathbf{R}$ is the number of leading 1's. Since these two numbers are the same, the row and column space have the same dimension. ■

Definition 2.1. (Rank/Nullity) Let $\mathbf{A} \in \mathbb{R}^{m\times n}$.

- The **rank** of $\mathbf{A}$, denoted by $\text{rank}(A)$, is the common dimension of the row space and column space of a matrix $\mathbf{A}$.
- The **nullity** of $\mathbf{A}$, denoted by $\text{nullity}(A)$, is the dimension of the null space of $\mathbf{A}$.

Remark: $\text{rank}(\mathbf{A}) \leq \min(m, n)$

Theorem 2.2. If $\mathbf{A}$ is a matrix with n columns, then

$$\text{rank}(\mathbf{A}) + \text{nullity}(\mathbf{A}) = n$$

Proof:

$$[\text{number of leading 1's variables}] + [\text{number of free variables}] = n$$

and $\text{rank}(\mathbf{A})$ =number of leading 1's variables, $\text{nullity}(\mathbf{A})$ = number of free variables. ■

Theorem 2.3. If $\mathbf{A}$ is an $m \times n$ matrix, then
(a) $\text{rank}(\mathbf{A})$ = the number of leading variables in the general solution of $\mathbf{Ax} = \mathbf{0}$.
(b) $\text{nullity}(\mathbf{A})$ = the number of parameters in the general solution of $\mathbf{Ax} = \mathbf{0}$.
(c) If $\mathbf{Ax} = \mathbf{b}$ is a consistent linear system, and if $\mathbf{A}$ has rank r, then the general solution of the system contains $n - r$ parameters.

Example 2.1. Find rank and nullity of the following matrix $\mathbf{A}$. Show that rank($\mathbf{A}$)+nullity($\mathbf{A}$)=n.

(a) $\mathbf{A} = \begin{bmatrix} 1 & -3 & 4 & -2 & 5 & 4 \\ 2 & -6 & 9 & -1 & 8 & 2 \\ 2 & -6 & 9 & -1 & 9 & 7 \\ -1 & 3 & -4 & 2 & -5 & -4 \end{bmatrix}$ and the corresponding RREF of $\mathbf{A}$ is $\mathbf{R} = \begin{bmatrix} 1 & -3 & 4 & -2 & 5 & 4 \\ 0 & 0 & 1 & 3 & -2 & -6 \\ 0 & 0 & 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$.

(b) Let $\mathbf{A} = \begin{bmatrix} 1 & 2 & 0 & 2 \\ -2 & -5 & 5 & 6 \\ 0 & -3 & 15 & 18 \\ 0 & -2 & 10 & 8 \\ 3 & 6 & 0 & 6 \end{bmatrix}$ and suppose the RREF of $\mathbf{A}$ is given by $\begin{bmatrix} 1 & 2 & 0 & 2 \\ 0 & 1 & -5 & -10 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$.

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Example 2.2. .
(a) Find the number of parameters in the general solution of $\mathbf{Ax} = 0$ if A is a 5×7 matrix of rank 3.
(b) Find the rank of a 5×7 matrix $\mathbf{A}$ for which $\mathbf{Ax} = 0$ has a two-dimensional solution space.

2.1 The Fundamental Spaces of a Matrix

There are six important vector spaces associated with a matrix $\mathbf{A}$ and its transpose $\mathbf{A}^T$:

- row($\mathbf{A}$), row($\mathbf{A}^T$)
- col($\mathbf{A}$), col($\mathbf{A}^T$)
- null($\mathbf{A}$), null($\mathbf{A}^T$)

Note that the row space of $\mathbf{A}^T$ is the same as the column space of $\mathbf{A}$, and the column space of $\mathbf{A}^T$ is the same as the row space of $\mathbf{A}$. Thus, of the six spaces listed above, only the following **four** are distinct:

- row($\mathbf{A}$), row($\mathbf{A}^T$)
- null($\mathbf{A}$), null($\mathbf{A}^T$)

and these are called the **fundamental spaces of a matrix A**.
We will now consider how these four subspaces are related.

Theorem 2.4. If $\mathbf{A}$ is any $m \times n$ matrix, then

- rank($\mathbf{A}$) = rank($\mathbf{A}^T$).
- rank($\mathbf{A}^T$) + nullity($\mathbf{A}^T$) = m
- rank($\mathbf{A}$) + nullity($\mathbf{A}^T$) = m

If rank(A) = r, then

$$\dim[\text{row}(\mathbf{A})] = r, \qquad \dim[\text{col}(\mathbf{A})] = r$$

$$\dim[\text{null}(\mathbf{A})] = n - r \quad \dim[\text{null}(\mathbf{A}^T)] = m - r.$$

Recall that if **A** is an $m \times n$ matrix, then the null space of **A** consists of those vectors that are orthogonal to each of the row vectors of **A**. To develop that idea in more detail, we make the following definition.

Definition 2.2. If W is a subspace of $\mathbb{R}^n$, then the set of all vectors in $\mathbb{R}^n$ that are orthogonal to every vector in W is called the orthogonal complement of W and is denoted by the symbol $W^\perp$.

The following theorem lists three basic properties of orthogonal complements.

Theorem 2.5. If W is a subspace of $\mathbb{R}^n$, then:
(a) $W^\perp$ is a subspace of $\mathbb{R}^n$.
(b) The only vector common to W and $W^\perp$is $\mathbf{0}$.
(c) The orthogonal complement of $W^\perp$ is W, i.e. $(W^\perp)^\perp = W$.

The next theorem will provide a geometric link between the fundamental spaces of a matrix.

Theorem 2.6. If **A** is an $m \times n$ matrix, then:
(a) The null space of **A** and the row space of **A** are orthogonal complements in $\mathbb{R}^n$.
(b) The null space of $\mathbf{A}^T$ and the column space of **A** are orthogonal complements in $\mathbb{R}^m$.

Recall that we have listed six results that are equivalent to the invertibility of a square matrix $\mathbf{A}$. We are now in a position to add ten more statements to that list to produce a single theorem that summarizes and links together all of the topics that we have covered thus far.

Theorem 2.7. Equivalent Statements:
If $\mathbf{A}$ is an $n \times n$ matrix, then the following statements are equivalent.

(a) $\mathbf{A}$ is invertible.

(b) $\mathbf{Ax} = \mathbf{0}$ has only the trivial solution.

(c) The reduced row echelon form of $\mathbf{A}$ is $\mathbf{I}_n$.

(d) $\mathbf{A}$ is expressible as a product of elementary matrices.

(e) $\mathbf{Ax} = \mathbf{b}$ is consistent for every $n \times 1$ matrix $\mathbf{b}$.

(f) $\mathbf{Ax} = \mathbf{b}$ has exactly one solution for every $n \times 1$ matrix $\mathbf{b}$.

(g) $\det(\mathbf{A}) \neq 0$.

(h) The column vectors of $\mathbf{A}$ are linearly independent.

(i) The row vectors of $\mathbf{A}$ are linearly independent.

(j) The column vectors of $\mathbf{A}$ span $\mathbb{R}^n$.

(k) The row vectors of $\mathbf{A}$ span $\mathbb{R}^n$.

(l) The column vectors of $\mathbf{A}$ form a basis for $\mathbb{R}^n$.

(m) The row vectors of $\mathbf{A}$ form a basis for $\mathbb{R}^n$.

(n) $\mathbf{A}$ has rank n.

(0) A has nullity $\mathbf{0}$.

(p) The orthogonal complement of the null space of $\mathbf{A}$ is $\mathbb{R}^n$.

(q) The orthogonal complement of the row space of $\mathbf{A}$ is $\{\mathbf{0}\}$.