# Differentiation

**Outline**

- Tangent line
- The Derivative
- Power and Sum Rules
- Product and Quotient Rules
- Derivatives of Trigonometric Functions
- Chain Rule
- Implicit Differentiation
- Derivatives of Inverse Functions
- Exponential & Logarithmic Functions
- Logarithmic Differentiation
- Derivatives of Hyperbolic Functions
- Linear Approximation and Differentials

## 1 Slope of Tangent Line

Suppose $f(x)$ is a continuous function. For a point $P$ on the graph $y = f(x)$, to find a line equation **tangent** to the graph $y = f(x)$ at $P$, it's mainly required to find the slope of this line.
For a linear function $f$, its graph is a straight line: $y = f(x)$ where $f(x) = mx + c$: for some constant $c$. Every point of graph for $y = f(x)$ has the same slope $m$ which can be computed from:

$$m = \frac{f(b) - f(a)}{b - a},$$

for any points $a, b$ on the domain of $f$. For general function $f(x)$, the slope $m$ of $f(x)$ at point $x = a$ can be approximated by using the slope of **secant line**, which is a straight line connecting the point $(a, f(a))$ and a near by point $(x, f(x))$, $x \neq a$. I.e.

$$m \approx \frac{f(x) - f(a)}{x - a}.$$

As the point $(x, f(x))$ used in the secant line gets closer to $(a, f(a))$, the approximation becomes more accurate. In particular, it can be obtained from the following definition.

**Definition 1.1** (Tangent Line)**.** Let $y = f(x)$ be continuous at the number $a$. If the limit

$$m = \lim_{x\to a} \frac{f(x) - f(a)}{x - a},$$

exits, then the tangent line to the graph of $f$ at $(a, f(a))$ is the line passing through the point $(a, f(a))$ with slope $m$.
Note that the slope $m$ can be written equivalently by using $x = a + h$, and

$$m = \lim_{h\to 0} \frac{f(a + h) - f(a)}{h}.$$

The following is a common procedure for finding the slop of the tangent line using the limit definition.

**Four-step process for computing** $m = \lim_{h \to 0} \frac{f(a+h)-f(a)}{h}$ To compute the slope $m = \lim_{h \to 0} \frac{f(a+h)-f(a)}{h}$, we can use the following 4 steps:

(1) Evaluate $f(a)$ and $f(a+h)$.

(2) Evaluate $f(a) - f(a+h)$ and simplify.

(3) Simplify the quotient

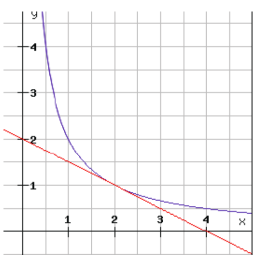
$$\frac{f(a+h)-f(a)}{h}.$$

(4) Compute the limit:

$$\lim_{h \to 0} \frac{f(a+h)-f(a)}{h}.$$

**Example 1.1.** Find the slope of the graph $f(x) = 2/x$ at $x = 2$ using the limit definition. Determine the tangent line equation of this graph at $x = 2$.

Solution:




Exercises: Find the slope of the graph $y = f(x)$ at $x = a$ and find an equation of the tangent line for $f(x)$ at $x = a$ for each of the following functions.

(a) $f(x) = x^2 + 2$, $a = 1$ (b) $f(x) = \sqrt{x-1}$, $a = 5$

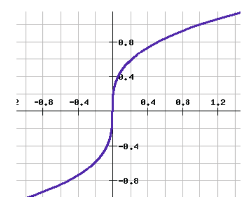
## 1.1 Types of Tangent Lines

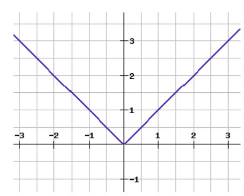
1. Tangent line with a constant slope
   $\Rightarrow$ A special case of this is a **horizontal tangent line** with slope $m = 0$.

2. **Vertical tangent line**: This case the slope at the point $(a, f(a))$ satisfies one of the following limits.
$$\lim_{h\to 0} \frac{f(a+h) - f(a)}{h} = \infty \qquad \text{or} \qquad \lim_{h\to 0} \frac{f(a+h) - f(a)}{h} = -\infty$$
   In this case the tangent line equation will be in the form $x = a$.

3. **A tangent line may not exist**. For a given point $(a, f(a))$, there is no tangent line at this point when
$$\lim_{h\to 0^-} \frac{f(a+h) - f(a)}{h} \neq \lim_{h\to 0^-} \frac{f(a+h) - f(a)}{h}.$$

**Example 1.2.** Find the slope of the following functions at $x = 0$ by using limit definition.
(a) $f(x) = x^{1/3}$ (b) $f(x) = |x|$
Solution:



# Rate of Change

We will consider *average* and *instantaneous* rate of change.

## 1.2 Average Rate of Change

**Average Velocity:** The average velocity or average speed is defined by

$$v_{avg} = \frac{\text{change of distance}}{\text{change in time}}.$$

For example, a runner who finishes a 10-km race in an elapsed time of 1 h 15 min (1.25 h) will have average velocity or average speed

$$v_{avg} = \frac{10-0}{1.25-0} = 8 \text{ km/h}.$$

Suppose the distance run in the time interval from 0 h to 0.5 h is measured to be 5 km. If we also determine that at 0.6 h the runner is 5.7 km from the starting line, then the average velocity during the time interval from 0.5 h to 0.6 h is given by

$$v_{avg} = \frac{5.7-5}{0.6-0.5} = 7 \text{ km/h}.$$

## 1.3 Instantaneous Rate of Change

**Definition 1.2** (Instantaneous Velocity)**.** Let $s = s(t)$ be a function that gives the position of an object moving in a straight line. Then the instantaneous velocity at time $t = t_0$ is

$$v(t_0) = \lim_{\triangle t \to 0} \frac{s(t_0 + \triangle t) - s(t_0)}{\triangle t} = \lim_{\triangle t \to 0} \frac{\triangle s}{\triangle t}. \quad (1)$$

**Example 1.3.** An object is moving in a straight line at time $t$ described by the function $s(t) = -t^2+6$. Find the instantaneous velocity of this object at $t = 2$.

## 2 The Derivative: Definition

**Definition 2.1.** The **derivative of** $f(x)$ **with respect to** $x$ , denoted by $f'(x)$, is defined as

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}, \tag{2}$$

whenever the limit exists.

**Example 2.1.** Find the derivative of $f(x) = x^3 + 2x^2$ by using the limit definition.
Solution:

**Example 2.2.** Find the derivative of $f(x) = \sqrt{x}$ by using the limit definition for general $x > 0$ and at $x = 1$.
Solution: [Hint: use *use rationalization*]

$$\begin{aligned} f(x+h) &= \sqrt{x+h} \\ \frac{f(x+h)-f(x)}{h} &= \end{aligned}$$

**Definition 2.2.** Let $f(x)$ be a function.

- **Right-hand derivative**

$$f'_+(x) = \lim_{h\to 0^+} \frac{f(x+h)-f(x)}{h}, \tag{3}$$

- **Left-hand derivative**

$$f'_-(x) = \lim_{h\to 0^-} \frac{f(x+h)-f(x)}{h}, \tag{4}$$

- **Differentiability**
  - ▷ $f(x)$ is is called differentiable at $x = a$ if $f'(x)$ exists.
  - ▷ $f(x)$ is differentiable on the open interval, e.g. $(a, b), (-\infty, b), (a, \infty)$, if the derivative exists for each point in that interval.
  - ▷ $f(x)$ is *differentiable everywhere* if it is differentiable at every point in $(-\infty, \infty)$.
  - ▷ $f(x)$ is differentiable on the closed interval $[a, b]$ if
    (i) $f$ is differentiable on $(a, b)$ and
    (ii) $f'_+(a)$ and $f'_-(b)$ both exist.
    .

**Example 2.3.** The graph of the function $f(x) = |x|$ does not have a tangent line at $(0, 0)$. Hence it is not differentiable at $x = 0$.

- For $x < 0$, $f(x) = |x| = -x$ and

  $f'(x) =$

- For $x > 0$, $f(x) = |x| = x$ and

  $f'(x) =$

- For $x = 0$, $f(0) = 0$ and

  $f'_-(0) =$

  and

  $f'_+(0) =$ .

  Since $f'_-(0) \neq f'_+(0)$, $f(x) = |x|$ is not differentiable at $x = 0$.

**Definition 2.3.** : Suppose $y = f(x)$ is continuous at $a$.

- $f(x)$ has a **horizontal tangent** at $(a, f(a))$ if $f'(a) = 0$.
- $f(x)$ has a **vertical tangent** at $(a, f(a))$ if $\lim_{x \to a} |f'(x)| = \infty$.

  Note that if $f$ is differentiable only on one side of $a$ (either from left or right), then it is only possible to have
  $$\lim_{x \to a^-} |f'(x)| = \infty \qquad \text{or} \lim_{\mathrm{x} \to \mathrm{a}^+} |\mathrm{f}'(\mathrm{x})| = \infty.$$
  In this case, we say that $f$ has a **one-sided vertical tangent** at $(a, f(a))$.
- A **normal line** at a point $(a, f(a))$ is the line that is *perpendicular* to the *tangent line* at that point.

**Example 2.4.** (One-sided vertical tangent) From Example 2.2, the derivative of $f(x) = \sqrt{x}$ is given by
$$f'(x) = \frac{1}{2\sqrt{x}}$$
for $x > 0$. The function $f(x) = \sqrt{x}$ is **not differentiable** on the interval $[0, \infty)$ because it is **not differentiable** at $x = 0$
$$f'_+(0) = \infty,$$
which implies that $f'_+(0)$ does not exist. However, we can find the **one-sided vertical tangent line** at $x = 0$ :
$$\lim_{x \to 0^+} |f'(x)| = \lim_{x \to 0^+} |\frac{1}{2\sqrt{x}}| = \infty.$$
Graph: ■

**Theorem 2.1** (Differentiability Implies Continuity)**.** If $f$ is **differentiable** at $a$, then $f$ is **continuous** at $a$

**Remark:** If $f$ is differentiable at $x = a$, then $f$ is continuous at $x = a$.
But $f(x)$ is continuous at $x = a$ **does not imply** that $f$ is differentiable at $x = a$.

**Example 2.5.** The functions $f(x) = |x|$ and $f(x) = x^{1/3}$ are **continuous** for all $x \in \mathbb{R}$. That is, it is continuous at $x = 0$. However, they are **not differentiable at** $x = 0$.

## 2.1 Remarks

- We often read $f'(x)$ as "$f$ prime of $x$."
- Other notations for derivative of $y = f(x)$ with respect to $x$:

$$f'(x) \qquad \frac{dy}{dx} \qquad y' \qquad Dy \qquad D_x y.$$

  Notations for derivative of $y = f(x)$ with respect to $x = a$:

$$f'(a) \qquad \left.\frac{dy}{dx}\right|_{x=a} \qquad y'(a) \qquad \left.D_x y\right|_{x=a}.$$

- We can think of derivative of function $f(x)$ at $x = a$ as the slope of $f(x)$ at $x = a$ and the equation of the tangent line is given by

$$y = f(a) + f'(a)(x - a).$$

- If $f$ is differentiable at $x = a$, then $f$ is continuous at $x = a$.
  But $f(x)$ is continuous at $x = a \not\Rightarrow f$ is differentiable at $x = a$.
- Notice that $f$ is not differentiable at $x = a$ if
  - (i) $f$ is discontinuous at $x = a$.
  - (ii) the graph of $f$ has a corner at $(a, f(a))$
  - (iii) the graph of $f$ has a vertical tangent line at $(a, f(a))$.

## 2.2 Examples

1. Let $f(x) = |x|$. Find $\lim_{x \to 0} f(x)$ and $f'(0)$.
2. Find the derivative $f'(x)$ of the following function using the definition of the derivative.
   - (a) $f(x) = 2x^2 + x + 1$.
   - (b) $f(x) = \sqrt{5x - 8}$.
3. Show that $f(x) = x^2$ has a horizontal slope at $x = 0$.

## 3 The Derivative: Power and Sum Rules

**Theorem 3.1** (Power Rule)**.** For any real number $n$,

$$\frac{d}{dx}x^n = nx^{n-1}.$$

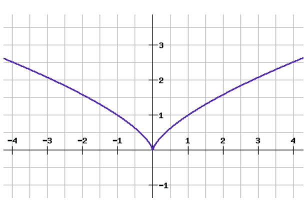
**Example 3.1.** Differentiate:

1. $x^{13}$
2. $x$
3. $x^{\sqrt{3}}$
4. $1$ ( i.e. $x^0$).

**Theorem 3.2** (Constant Function)**.** If $f(x) = c$ is a constant function, then $f'(x) = 0$.

**Proof:** (Exercise)By the limit definition of derivative,.....

**Example 3.2.** Differentiate $y = x^{2/3}$. Find the tangent line equation of the graph $y = x^{2/3}$ at $x = 0$.
Solution:




Note that $y = x^{2/3}$ is continuous at $x = 0$, but it's not differentiable at $x = 0$. Note also that $\lim_{x\to 0^+} f'(x) = \infty$ and $\lim_{x\to 0^-} f'(x) = -\infty$. That is,

$$\lim_{x\to 0} |f'(x)| =$$

and the tangent line is the $y$-axis itself, or the line equation is ...................

### Higher-Order Derivatives

- The derivative of a function $f$ is sometimes called **the first derivative** of $f$.
- The **second** derivative of a function $f$ can be defined by taking derivative of $f'(x) = \frac{dy}{dx}$. I.e., the second derivative of $y = f(x)$ is

$$\frac{d}{dx}\left(\frac{dy}{dx}\right).$$

Other notations for the second derivative of $y = f(x)$ with respect to $x$ are

$$f''(x) \qquad \frac{d^2y}{dx^2} \qquad \frac{d^2}{dx^2}f(x) \qquad y'' \qquad D_x^2 \qquad D^2.$$

- The **third** derivative of a function $f$ can be defined by taking derivative of $f''(x) = \frac{d^2y}{dx^2}$. I.e., the third derivative of $y = f(x)$ is

$$\frac{d}{dx}\left(\frac{d^2y}{dx^2}\right).$$

Similarly, other notations for the third derivative of $y = f(x)$ with respect to $x$ are

$$f'''(x) \qquad \frac{d^3y}{dx^3} \qquad \frac{d^3}{dx^3}f(x) \qquad y''' \qquad D_x^3 \qquad D^3.$$

**Definition 3.1.** The $n$**-th** derivative of a function $f$ is defined by taking derivative of the $(n-1)$**-th** derivative $f^{(n-1)}(x)$, for $n = 1, 2, 3, \ldots$. I.e., the $n$-th derivative of $y = f(x)$ with respect to $x$ is

$$\frac{d^ny}{dx^n} = \frac{d}{dx}\left(\frac{d^{(n-1)}y}{dx^{(n-1)}}\right).$$

Similarly, other notations for the $n$-th derivative of $y = f(x)$ with respect to $x$ are

$$f^{(n)}(x) \qquad \frac{d^ny}{dx^n} \qquad \frac{d^n}{dx^n}f(x) \qquad y^{(n)} \qquad D_x^n \qquad D^n.$$

**Example 3.3.** Let $f(x) = x^{10}$.
(a) Find the second and the fourth derivatives of $f(x)$.
(b) Find the derivative: $f^{(11)}$.

**Theorem 3.3** (Constant Multiple Rule)**.** If $c$ is a constant and $f$ is a differentiable function at $x$, then $cf$ is a differentiable function at $x$ and then $\frac{d}{dx}cf(x) = c\frac{df(x)}{dx} = cf'(x)$.

**Theorem 3.4** (Sum and Difference Rules)**.** Let $f$ and $g$ be differentiable functions at $x$. Then $f + g$ and $f - g$ are differentiable at $x$ and

$$(f(x) + g(x))' = f'(x) + g'(x)$$

$$(f(x) - g(x))' = f'(x) - g'(x).$$

**Example 3.4.** Differentiate
(a) $y = -3\sqrt[3]{x^7} + e^{\pi} + \frac{4}{x\sqrt{x}} + 9$
(b) $y = \frac{x^5 - x^{3/2} + 1}{\sqrt{x}}$
(c) $y = \frac{x^2+3x+2}{x+1}$
.

# 4 The Derivative: Product and Quotient Rules

**Theorem 4.1** (Product Rule)**.** Let $f$ and $g$ be differentiable functions at $x$. Then the product $fg$ is differentiable at $x$ and

$$\frac{d}{dx}[f(x)g(x)] = f(x)\frac{d}{dx}g(x) + g(x)\frac{d}{dx}f(x).$$

**Example 4.1.** (a) Differentiate $y = (2 + \sqrt{x} + x)(x^3 - 3x + 1)$.
(b) Find the tangent line to the graph $y = (2 + \sqrt{x} + x)(x^3 - 3x + 1)$ at $x = 1$.

**Example 4.2.** Differentiate $y = (2 + x)(x^3 + 1)(x^2 - 7x)$.

**Theorem 4.2** (Quotient Rule)**.** Let $f$ and $g$ be *differentiable* functions at $x$ and $g(x) \neq 0$. Then $f/g$ is differentiable at $x$ and

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{g(x)f'(x) - f(x)g'(x)}{[g(x)]^2}.$$

**Example 4.3.** Differentiate

$$y = \frac{2 + x\sqrt{x}}{x^3 + 1}.$$

**Example 4.4.** Differentiate

$$y = \frac{2 + x}{(x^3 + 1)(x^2 - 7x)}.$$

**Example 4.5.** Let

$$h(x) = \frac{x^2 + 1}{g(x)}.$$

where $g(x)$ is a non-zero function with $g(1) = 2$ and $g'(1) = 4$. Find an equation of the tangent line to the graph of $y = h(x)$ at $x = 1$.

# 5 Derivatives of Trigonometric Functions

## 5.1 Derivatives of Sine and Cosine

**Theorem 5.1.** Let $x$ be any real number. Then

$$\frac{d}{dx}\sin(x) = \cos(x), \qquad \frac{d}{dx}\cos(x) = -\sin(x).$$

**Proof:** From the limit definition for derivative, the derivative of function $f$ at $x$ is given by

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}. \tag{5}$$

Then, if we let $f(x) = \sin(x)$, and use the identity

$$\sin(x+h) = \sin(x)\cos(h) + \sin(h)\cos(x),$$

$$\begin{aligned}
\frac{d}{dx}\sin(x) &= \lim_{h\to 0}\frac{\sin(x+h) - \sin(x)}{h} \\
&= \lim_{h\to 0}\frac{[\sin(x)\cos(h) + \sin(h)\cos(x)] - \sin(x)}{h} \\
&= \lim_{h\to 0}\frac{\sin(x)[\cos(h) - 1] + \sin(h)\cos(x)}{h} \\
&= \lim_{h\to 0}\frac{\sin(x)[\cos(h) - 1]}{h} + \lim_{h\to 0}\frac{\sin(h)\cos(x)}{h} \\
&= \sin(x)\underbrace{\lim_{h\to 0}\frac{[\cos(h) - 1]}{h}}_{=0 \text{ from } (6)} + \cos(x)\underbrace{\lim_{h\to 0}\frac{\sin(h)}{h}}_{=1 \text{ from } (6)} \\
&= \cos(x).
\end{aligned}$$

Similarly, we can use the limit definition of derivative (5) to show that $\frac{d}{dx}\cos(x) = -\sin(x)$. Note that above used the following facts:

$$\lim_{h\to 0}\frac{\sin(h)}{h} = 1, \qquad \lim_{h\to 0}\frac{[\cos(h) - 1]}{h} = 0. \tag{6}$$

## 5.2 Derivatives of Tangent and Cotangent

To find the derivative of $\tan(x)$, we use the fact that $\tan(x) = \frac{\sin(x)}{\cos(x)}$ and then apply the *quotient rule*:

$$\frac{d}{dx}\tan(x) = \frac{d}{dx}\left(\frac{\sin(x)}{\cos(x)}\right) = \frac{\cos(x)\frac{d}{dx}\sin(x) - \sin(x)\frac{d}{dx}\cos(x)}{\cos^2(x)} = \frac{\cos^2(x) + \sin^2(x)}{\cos^2(x)} = \sec^2(x).$$

Note that, we have used the identity $\cos^2(x) + \sin^2(x) = 1$ and $\sec(x) = \frac{1}{\cos(x)}$. Similarly, we can obtain the derivative of cotangent:

$$\frac{d}{dx}\cot(x) = -\csc^2(x). \tag{7}$$

## 5.3 Derivative of Secant and Cosecant

To find the derivative of secant and cosecant, we can use the fact that $\sec(x) = \frac{1}{\cos(x)}$, $\csc(x) = \frac{1}{\sin(x)}$, with the *quotient rule* and $\frac{d}{dx}\sin(x) = \cos(x)$, $\frac{d}{dx}\cos(x) = -\sin(x)$. Then

$$\frac{d}{dx}\sec(x) = \frac{d}{dx}\left(\frac{1}{\cos(x)}\right) = \frac{\cos(x)\frac{d}{dx}1 - 1\frac{d}{dx}\cos(x)}{\cos^2(x)} = \frac{0+\sin(x)}{\cos^2(x)} = \sec(x)\tan(x),$$

and

$$\frac{d}{dx}\csc(x) = \frac{d}{dx}\left(\frac{1}{\sin(x)}\right) = \frac{\sin(x)\frac{d}{dx}1 - 1\frac{d}{dx}\sin(x)}{\sin^2(x)} = \frac{0-\cos(x)}{\sin^2(x)} = -\csc(x)\cot(x).$$

**Theorem 5.2 (Summary: Derivatives of Trigonometric Functions).** :

| | |
|---|---|
| $\frac{d}{dx}\sin(x) = \cos(x)$ | $\frac{d}{dx}\cos(x) = -\sin(x)$ |
| $\frac{d}{dx}\tan(x) = \sec^2(x)$ | $\frac{d}{dx}\cot(x) = -\csc^2(x)$ |
| $\frac{d}{dx}\sec(x) = \sec(x)\tan(x)$ | $\frac{d}{dx}\csc(x) = -\csc(x)\cot(x)$ |

**Example 5.1.** Find the first derivatives of the following functions.
(a) $f(x) = 3\sec(x) - 10\cot(x)$
(b) $f(x) = \frac{\sin(x)+x\tan(x)}{3-2\cos(x)}$

**Example 5.2.** Let $f(x) = x\sin(x)$. Find $f''(x)$. [Ans:$-x\sin(x) + 2\cos(x)$]

**Example 5.3.** (Exercise) Let $f(x) = \sin(x)$. Find the $n$-th derivative of $f$, $f^{(n)}(x)$, for $n = 4, 18, 100$.

# 6 Chain Rule

**Theorem 6.1. (Chain Rule):** Let $f(x)$ and $g(x)$ be differentiable functions. Consider the composite function

$$F(x) = (f \circ g)(x) = f(g(x)).$$

Then the function $F(x)$ is differentiable and

$$\boxed{F'(x) = \tfrac{d}{dx}[f(g(x))] = f'(g(x))g'(x)}. \tag{8}$$

Alternatively,

$$\boxed{y = f(u) \text{ and } u = g(x) \quad \Longrightarrow \quad \frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}}. \tag{9}$$

Note that $y = f(u) = f(g(x)) = F(x)$ and therefore the derivative of $F(x)$: $\frac{d}{dx}F(x) = \frac{dy}{dx}$.

## 6.1 Special Cases:

**Power Rule for Functions:**

Let $n$ be any real number. Suppose $u = g(x)$ is differentiable at $x$. Then

$$\boxed{\frac{d}{dx}[g(x)]^n = n[g(x)]^{n-1}\frac{d}{dx}g(x)}. \tag{10}$$

Equivalent,

$$\boxed{\frac{d}{dx}u^n = nu^{n-1}\frac{du}{dx}}. \tag{11}$$

**Example 6.1.** Differentiate $F(x) = (1 + 3x + 5x^5 + 4x^{10})^7$.

**Example 6.2.** Differentiate
(a) $F(x) = \sin^{10}(2x+1)$
(b) $F(x) = \big(x^3 + \sin^3(x)\big)^{100}$.

**Derivatives of Trigonometric Functions:**

Suppose $u = g(x)$ is differentiable at $x$. Then

| | |
|---|---|
| $\frac{d}{dx}\sin(u) = \cos(u)\frac{du}{dx}$ | $\frac{d}{dx}\cos(u) = -\sin(u)\frac{du}{dx}$ |
| $\frac{d}{dx}\tan(u) = \sec^2(u)\frac{du}{dx}$ | $\frac{d}{dx}\cot(u) = -\csc^2(u)\frac{du}{dx}$ |
| $\frac{d}{dx}\sec(u) = \sec(u)\tan(u)\frac{du}{dx}$ | $\frac{d}{dx}\csc(u) = -\csc(u)\cot(u)\frac{du}{dx}$. |

**Example 6.3.** Differentiate $F(x) = \cos^3(\sin(x))$.

**Example 6.4** (Exercise)**.** Differentiate

$$y = \left(\frac{\sqrt{x + \tan(\pi x/2)}}{\sqrt[3]{4x^2+1} + \ln 2}\right)^{3/5}$$

$$y = \cos^3\left(\frac{x^3}{x^3 + \sin(6x)}\right).$$

# 7 Implicit Differentiation

## 7.1 Implicit and explicit functions

**Definition 7.1.** Let $x$ be the *independent variable* and $y$ be the *dependent variable* .

- An *explicit function* $f$ is a function in which the dependent variable $y$ is solely expressed in terms of the independent variable $x$, i.e. it is written in the form: $y = f(x)$.
- An *implicit function* $f$ is a function that is not written in the form $y = f(x)$, but it is defined through an equation $F(x, y) = 0$ on some interval $\mathcal{I}$, i.e. $F(x, f(x)) = 0$ for some $x \in \mathcal{I}$.

**Example 7.1.** Implicit and explicit functions.

- $y = x^3 + 1$, $y = \sqrt{2x - 1}$ are explicit functions.
- $x^2 + y^2 = 2$ and $x^3 + y^2 + xy = 0$ are implicit functions.
- In general, we may not be able to write $F(x, y) = 0$ in the form of an explicit function $f(x)$. For example, we cannot write $y$ in term of $x$ for the equation: $x^4 + x^2y^3 - 2x - \sin(y) = y^5$.

We can compute the derivatives $\frac{dy}{dx}$ for the function defined implicitly by using the following steps.

**Steps for computing $dy/dx$ using implicit differentiation:**

$(i)$ Differentiate with respect to $x$ throughout the equation, or differentiate both sides of the equation.

$(ii)$ Solve for $\frac{dy}{dx}$ in terms of independent variable $x$ and dependent variable $y$.

**Example 7.2.** Consider $x^2 + y^2 = 2$.

(i) Find $\frac{dy}{dx}$ by using *implicit differentiation.*

(ii) Solve for $y$ explicitly in terms of $x$ and differentiate to find $\frac{dy}{dx}$.

(iii) Show that the answers in (i) and (ii) are equivalent.

Solution:(i) To find $\frac{dy}{dx}$, we follow the above two steps.

(i) Differentiate both sides of the equation:

$$\begin{aligned} \frac{d}{dx}\left(x^2 + y^2\right) &= \frac{d}{dx}2 \\ 2x\underbrace{\frac{d}{dx}x}_{=1} + 2y\frac{d}{dx}y &= 0 \\ 2x + 2y\frac{dy}{dx} &= 0 \end{aligned}$$

(ii) Solve for $\frac{dy}{dx}$ in terms of $x$ and $y$: from (i)

$$2x + 2y\frac{dy}{dx} = 0 \implies 2y\frac{dy}{dx} = -2x \implies \frac{dy}{dx} = -2x/2y = -x/y.$$

Therefore, $\frac{dy}{dx} = -\frac{x}{y}$. □

Solution:(ii) Consider the graph $x^2 + y^2 = 2$. We can write $y$ in terms of $x$ explicitly as:

$$y = \pm\sqrt{2 - x^2}.$$

Hence, this can be written as two separate explicit functions:

$$f(x) = \sqrt{2 - x^2} \qquad \text{and} \qquad g(x) = -\sqrt{2 - x^2}.$$

The derivative $\frac{dy}{dx}$ therefore depends on the value of $y$. In particular,

- for $y > 0$, $\frac{dy}{dx} = \frac{d\,f(x)}{dx} = \frac{d}{dx}\left(\sqrt{2 - x^2}\right) = \frac{1}{2}(2 - x^2)^{-1/2} \cdot (-2x) = -\frac{1}{\sqrt{2-x^2}}$
- for $y < 0$, $\frac{dy}{dx} = \frac{dg(x)}{dx} = \frac{d}{dx}\left(-\sqrt{2 - x^2}\right) = -\frac{1}{2}(2 - x^2)^{-1/2} \cdot (-2x) = \frac{1}{\sqrt{2-x^2}}$.

Solution:(iii) It can be shown that the answers in (i) and (ii) are equivalent by substituting $y = \pm\sqrt{2 - x^2}$ into the answer in (i). ■

**Example 7.3.** Find the slopes for the tangent lines to the graph $x^2+y^2 = 2$ at the points $(x, y) = (1, 1)$ and $(1, -1)$

Solution:
At $(x, y) = (1, 1)$, the slope is given by the derivative $\frac{dy}{dx}$ evaluate at $(x, y) = (1, 1)$: from the previous example,

$$\left.\frac{dy}{dx}\right|_{(x,y)=(1,1)} = -\left.\frac{x}{y}\right|_{(x,y)=(1,1)} = -1.$$

That is, the slope of the graph $x^2 + y^2 = 2$ at the point $(x, y) = (1, 1)$ is $-1$. The equation of the tangent line can be obtained from

$$-1 = \frac{y - 1}{x - 1} \implies y - 1 = -1(x - 1) \implies y = -x + 2.$$

At $(x, y) = (1, -1)$, the slope is given by the derivative $\frac{dy}{dx}$ evaluate at $(x, y) = (1, -1)$: from the previous example,

$$\left.\frac{dy}{dx}\right|_{(x,y)=(1,-1)} = -\left.\frac{x}{y}\right|_{(x,y)=(1,-1)} = 1.$$

That is, the slope of the graph $x^2 + y^2 = 2$ at the point $(x, y) = (1, -1)$ is 1. The equation of the tangent line can be obtained from

$$1 = \frac{y - (-1)}{x - 1} \implies y + 1 = 1(x - 1) \implies y = x - 2.$$

■

**Example 7.4** (Exercise)**.** Consider $x^2 + y^2 = 4$.
Find $\frac{dy}{dx}$. Then, find the slope of the tangent line on the graph of $x^2+y^2 = 4$ at the points corresponding to $x = 1$. [ANS: (a) $-x/y$, (b) at $(1, -\sqrt{3}) : 1/\sqrt{3}$, at $(1, \sqrt{3}) : -1/\sqrt{3}$]

**Example 7.5** (Exercise)**.** Find $y'(x)$ if $\sin(y) = y\cos(2x)$. [ANS: $y'(x) = -(2y\sin(2x))/(\cos(y) - \cos(2x))$]

**Example 7.6.** Find $dy/dx$ if $x^4 + x^2y^3 - y^5 = 2x + 1$. [ANS: $dy/dx = (2 - 4x^3 - 2xy^3)/(3x^2y^2 - 5y^4)$]

**Example 7.7.** Find an equation of the tangent line to the graph $\cos(xy^2) = y^2 + x$ at the point $(0, 1)$

### 7.2 Implicit Differentiation for Higher-order derivatives

Recall that the second-order derivative can be obtain from the first derivative:

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right).$$

Similarly, the $n$-th derivative can be defined from the order $(n-1)$-th derivative:

$$\frac{d^ny}{dx^n} = \frac{d}{dx}\left(\frac{d^{(n-1)}y}{dx^{(n-1)}}\right),$$

for $n = 2, 3, \ldots$. We can find higher derivatives using the above recursive definition.

**Example 7.8.** Consider again $x^2 + y^2 = 2$. Find the second derivative $\frac{d^2y}{dx^2}$ and the third derivative $\frac{d^3y}{dx^3}$ by using the implicit differentiation.
Ans.

$$\frac{d}{dx}\left(x^2+y^2\right) = \frac{d}{dx}2 \quad\Longrightarrow\quad 2x + 2y\frac{dy}{dx} = 0 \quad\Longrightarrow\quad \frac{dy}{dx} = -\frac{x}{y}.$$

Using the recursive definition above:

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left(-\frac{x}{y}\right) = -\frac{y\frac{dx}{dx} - x\frac{dy}{dx}}{y^2} = -\frac{y - x\left(-\frac{x}{y}\right)}{y^2} = -\frac{\frac{y^2+x^2}{y}}{y^2} = -\frac{y^2+x^2}{y^3} = -\frac{2}{y^3},$$

where we have used the fact that $y^2 + x^2 = 2$, and

$$\frac{d^3y}{dx^3} = \frac{d}{dx}\left(\frac{d^2y}{dx^2}\right) = \frac{d}{dx}\left(-\frac{2}{y^3}\right) = -\frac{y^3\frac{d}{dx}2 - 2\frac{dy^3}{dx}}{y^6} = -\frac{0 - 2\cdot 3y^2\frac{dy}{dx}}{y^6} = \frac{6y^2}{y^6}\left(\frac{dy}{dx}\right) = \frac{6}{y^4}\left(-\frac{x}{y}\right) = -\frac{6x}{y^5}$$

where we have used the fact that $\frac{dy}{dx} = -\frac{x}{y}$. ■

**Example 7.9.** Find $d^2y/dx^2$ if $x^2 + y^2 = 4$. [ANS: $d^2y/dx^2 = -4/y^3$] —Exercise.

**Example 7.10.** Find $\left.\frac{d^2y}{dx^2}\right|_{(x,y)=(0,1)}$ if $y^4 - xy = 4$. [ANS: -1/16]

# 8 Derivatives of Inverse Functions

**Definition 8.1** (Inverse Function)**.** Let $f$ be a one-to-one function with domain $X$ and range $Y$. The inverse function $f^{-1}$ with domain $Y$ and range $X$ is the function such that

$$f(f^{-1}(x)) = x, \quad \text{for every } x \in Y \qquad \text{and} \qquad f^{-1}(f(x)) = x, \quad \text{for every } x \in X.$$

For a one-to-one function $f$, the graph of $f$ and its inverse function $f^{-1}$ are reflections of each other in the line $y = x$ and if a point $(a, b)$ is on $f$, then $(b, a)$ is on $f^{-1}$. We will see that the **slopes of tangent lines** to the graph of a differentiable function $f$ are related to the **slopes of tangent lines** to the graph of $f^{-1}$.

**Example 8.1.** Consider $f(x) = x^2 + 1$ for $x \geq 0$. Find $f^{-1}(x)$, $(f^{-1})'(x)$, and $(f^{-1})'(5)$.

**Theorem 8.1** (Derivative of an Inverse Function )**.** Suppose

$$\left\{ \begin{array}{ll} (i) & f \text{ is differentiable} \\ (ii) & f'(x) \neq 0 \\ (iii) & f^{-1} \text{exists} \end{array} \right\}$$

for all $x \in \mathcal{I}$, where $\mathcal{I}$ is an open interval. Then $f^{-1}$ is differentiable and

$$\frac{d}{dx} f^{-1}(x) = \frac{1}{f'(f^{-1}(x))}.$$

**Example 8.2.** Consider $f(x) = x^2 + 1$ for $x \geq 0$. Use the theorem above to find $(f^{-1})'(5)$.

**Example 8.3.** (Derivative of Inverse using Implicit Differentiation)
Consider the function $f(x) = 5x^3 + 8x - 9$. Find the slope of the inverse function $f^{-1}(x)$ at $(4, 1)$.
Solution: Notice that $f(1) = 4$. That is, $(1, 4)$ is on $f(x)$ and so we must have $(4, 1)$ is on the graph $f^{-1}(x)$ We can find inverse by considering this as

$$y = 5x^3 + 8x - 9$$

and we switching $x$ and $y$ to find its inverse:

$$x = 5y^3 + 8y - 9.$$

That is $\frac{d}{dx}f^{-1}(x) = \frac{dy}{dx}$ for the above equation. Using implicit differentiation we have

$$\begin{aligned} \frac{d}{dx}x &= \frac{d}{dx}(5y^3 + 8y - 9) \\ 1 &= 15y^2\frac{dy}{dx} + 8\frac{dy}{dx} \\ \frac{d}{dx}f^{-1}(x) = \frac{dy}{dx} &= \frac{1}{15y^2 + 8} \end{aligned}$$

and therefore the slope of the inverse function at $(4, 1)$ is

$$\frac{d}{dx}f^{-1}(x)\bigg|_{(x,y)=(4,1)} = \frac{dy}{dx}\bigg|_{(x,y)=(4,1)} = \frac{1}{15y^2 + 8}\bigg|_{(x,y)=(4,1)} = \frac{1}{14(1) + 8} = \frac{1}{23}.$$

Notice that we can find $\frac{d}{dx}f^{-1}(x)\big|_{(x,y)=(4,1)}$ without actually computing $f^{-1}(x)$. ■

**Example 8.4.** Find the derivative of $f(x) = \sin^{-1}(x)$.
Solution: Since $y = \sin^{-1}(x)$ is the inverse function of sin, we have

$$x = \sin(y)$$

and we can find $\frac{dy}{dx}$ of the above equation by using implicit differentiation:

$$\frac{dx}{dx} = \frac{d}{dx}\sin(y) \quad \Rightarrow \quad 1 = \cos(y)\frac{dy}{dx} \quad \Rightarrow \quad \frac{dy}{dx} = \frac{1}{\cos(y)}. \tag{12}$$

From the trigonometric identity: $\cos^2(y) + \sin^2(y) = 1$, using $x = \sin(y)$ gives

$$\cos(y) = \sqrt{1 - \sin^2(y)} = \sqrt{1 - x^2}. \tag{13}$$

From (12) and (13),

$$\boxed{\frac{d}{dx}\sin^{-1}(x) = \frac{1}{\sqrt{1 - x^2}}.}$$

■

**Derivatives of Inverse Trigonometric Functions:**

Suppose $u = g(x)$ is differentiable at $x$. Then

$$\frac{d}{dx}\sin^{-1}(u) = \frac{1}{\sqrt{1-u^2}}\frac{du}{dx} \qquad \frac{d}{dx}\cos^{-1}(u) = \frac{-1}{\sqrt{1-u^2}}\frac{du}{dx}, \quad |u| < 1$$

$$\frac{d}{dx}\tan^{-1}(u) = \frac{1}{1+u^2}\frac{du}{dx} \qquad \frac{d}{dx}\cot^{-1}(u) = \frac{-1}{1+u^2}\frac{du}{dx}$$

$$\frac{d}{dx}\sec^{-1}(u) = \frac{1}{|u|\sqrt{u^2-1}}\frac{du}{dx} \qquad \frac{d}{dx}\csc^{-1}(u) = \frac{-1}{|u|\sqrt{u^2-1}}\frac{du}{dx}, \quad |u| > 1.$$

**Example 8.5.** Differentiate
(a) $y = \sin^{-1}(2x) + \cos^{-1}(\sqrt{x+2})$
(b) $y = \sec^{-1}(x^2 + 1)$ [Exercise].

**Example 8.6.** Find the tangent line equation of $y = \arctan(2x) + 1$ at $(0, 1)$.

# 9 Derivatives of Exponential & Logarithmic Functions Exponential Functions

**Theorem 9.1** (Derivative of Exponential Functions)**.** Let $u = g(x)$ be a differentiable function. Then,

$$\frac{d}{dx}e^u = e^u \frac{du}{dx} \tag{14}$$

$$\frac{d}{dx}b^u = b^u \ \ln(b) \ \frac{du}{dx} \tag{15}$$

where $b$ is a constant positive real number.

Note that, to prove the above theorem, we can use the fact that the natural number $e$ is given by

$$e = \lim_{h\to 0}(1+h)^{1/h}$$

and so $1 + h \to e^h$ as $h \to 0$. That is,

$$\lim_{h\to 0}\frac{e^h - 1}{h} = 1.$$

Let $f(x) = e^x$. Then, by the limit definition of the derivative:

$$\frac{d}{dx}e^x = \lim_{h\to 0}\frac{f(x+h) - f(x)}{h} =$$

■

Remark: To prove $\frac{d}{dx}b^x$, use the fact that

$$f(x) = b^x = \left(e^{\ln(b)}\right)^x = e^{\ln(b)\ x}.$$

**Example 9.1.** Differentiate (a) $y = e^{-x^2}$ (b) $y = 10^{3/x^3}$.

**Theorem 9.2** (Derivative of Logarithmic Functions)**.** Let $u = g(x)$ be a differentiable function. Then,

$$\frac{d}{dx}\ln(u) = \frac{1}{u}\frac{du}{dx} \tag{16}$$

$$\frac{d}{dx}\log_b(u) = \frac{1}{u\ \ln(b)}\frac{du}{dx} \tag{17}$$

where $b$ is a constant positive real number.

To prove this, the implicit differentiation can be used here: $y = \ln(x)$ if $x = e^y$ and

$$\frac{dy}{dx}x = \frac{d}{dx}e^y \quad \Rightarrow \quad 1 = e^y\frac{dy}{dx} \quad \Rightarrow \quad \frac{dy}{dx} = \frac{1}{e^y} = \frac{1}{x} \quad \Rightarrow \quad \frac{d}{dx}\ln(x) = \frac{1}{x}.$$

**Example 9.2.** Differentiate (a) $f(x) = \ln(\tan(x))$ (b) $y = \ln(\ln(x))$ (c) $y = \sin^{-1}(\log_2(3x))$
.

Let $u = g(x)$ be a differentiable function. Then

$$\frac{d}{dx}\ln(|u|) = \frac{1}{u}\frac{du}{dx}. \tag{18}$$

To prove $\boxed{\frac{d}{dx}\ln(|x|) = \frac{1}{x}}$ for $x \neq 0$, we consider 2 cases.
(i) For $x > 0$, $\frac{d}{dx}\ln(|x|) = \frac{d}{dx}\ln(x) = \frac{1}{x}$.
(ii) For $x < 0$, $\frac{d}{dx}\ln(|x|) = \frac{d}{dx}\ln(-x) = \frac{1}{-x}\frac{d(-x)}{dx} = \frac{1}{x}$.

**Example 9.3.** Differentiate $y = \ln(|2x + 5|)$.

# 10 Logarithmic Differentiation

We next look at how to differentiate a function where both the base and the exponent are variable, e.g. $f(x) = x^{\sqrt{x}}$, or $f(x) = (1 + 1/x)^x$.

**Steps of Logarithmic Differentiation**

(i) Take the natural logarithm of both sides of $y = f(x)$:

$$\ln(y) = \ln(f(x))$$

Simplify the right-hand side of above equation using the properties of logarithms.

(ii) Differentiate the simplify version from (i) using **implicit differentiation**:

$$\frac{d}{dx}\ln(y) = \frac{d}{dx}\ln(f(x)).$$

(iii) Use $\frac{d}{dx}\ln(y) = \frac{1}{y}\frac{dy}{dx}$ , solve for $\frac{dy}{dx}$, and replace $y$ by $f(x)$:

$$f'(x) = \frac{dy}{dx} = f(x)\frac{d}{dx}\ln f(x).$$

**Example 10.1.** Differentiate $f(x) = x^x$.

**Example 10.2** (Simplifying before differentiating)**.** Differentiate $y = \ln\left(\frac{x^2}{\sqrt{(x-1)^3(2x+1)^7}}\right)$.

**Example 10.3.** Let $a, b, c, d$ be constant positive integers. Differentiate

$$f(x) = \frac{(x-2)^a(x-3)^b}{(x+4)^c(x+5)^d}.$$

**Example 10.4.** Find the tangent line to the graph of $y = x(\ln(x))^x$ at $x = e$.

**Example 10.5.** (Exercise) Differentiate

$$y = \frac{(\sin(x) + 1)^x}{x^3}.$$

# 11 Derivatives of Hyperbolic Functions

**Definition 11.1.** For any real number $x$, the **hyperbolic sine** of $x$ is

$$\sinh(x) = \frac{e^x - e^{-x}}{2}$$

and the **hyperbolic cosine** of $x$ is

$$\cosh(x) = \frac{e^x + e^{-x}}{2}.$$

Remarks:

- The domain for each of $\sinh(x)$ and $\cosh(x)$ is the set of real numbers $\mathbb{R}$, due to their definitions that are based on exponential functions.
- By using the definitions, $\sinh(0) = 0$ and $\cosh(0) = 1$.
- It is possible to define other **hyperbolic functions** as follows.

  - Hyperbolic tangent of $x$: $\tanh(x) = \frac{\sinh(x)}{\cosh(x)} = \frac{e^x - e^{-x}}{e^x + e^{-x}}$
  - Hyperbolic cotangent of $x$, $x \neq 0$: $\coth(x) = \frac{\cosh(x)}{\sinh(x)} = \frac{e^x + e^{-x}}{e^x - e^{-x}}$
  - Hyperbolic secant of $x$, : $\text{sech}(x) = \frac{1}{\cosh(x)} = \frac{1}{e^x + e^{-x}}$
  - Hyperbolic cosecant of $x$, $x \neq 0$: $\text{csch}(x) = \frac{1}{\sinh(x)} = \frac{1}{e^x - e^{-x}}$

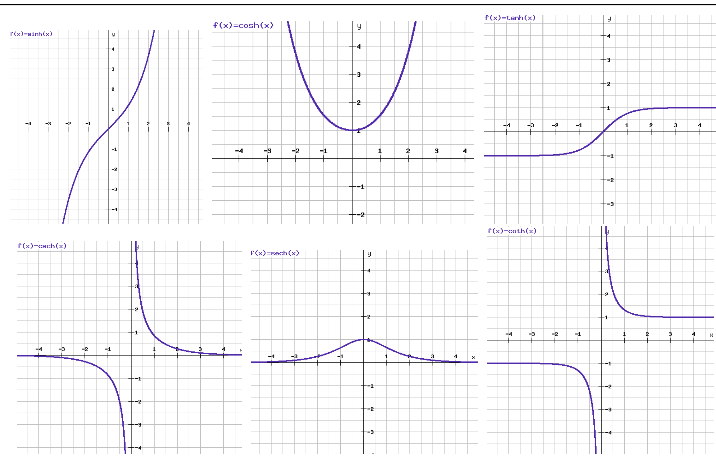


Figure 1: Hyperbolic functions: $\sinh(x)$, $\cosh(x)$, $\tanh(x)$, $\text{csch}(x)$, $\text{sech}(x)$, $\coth(x)$.

The following are the identities for hyperbolic functions.

| | |
|---|---|
| $\sinh(-x) = -\sinh(x)$ | $\sinh(x+y) = \sinh(x)\cosh(x) + \cosh(x)\sinh(x)$ |
| $\cosh(-x) = cosh(x)$ | $\sinh(x-y) = \sinh(x)\cosh(x) - \cosh(x)\sinh(x)$ |
| $\tanh(-x) = -\tanh(x)$ | $\cosh(x+y) = \cosh(x)\cosh(x) + \sinh(x)\sinh(x)$ |
| $\cosh^2(x) - \sinh^2(x) = 1$ | $\cosh(x-y) = \cosh(x)\cosh(x) - \sinh(x)\sinh(x)$ |
| $1 - \tanh^2(x) = \text{sech}^2(x)$ | $\sinh(2x) = 2\sinh(x)\cosh(x)$ |
| $\coth^2(x) - 1 = \text{csch}^2(x)$ | $\cosh(2x) = \cosh^2(x) + \sinh^2(x)$ |
| $\sinh^2(x) = \frac{1}{2}(-1 + \cosh(2x))$ | $\cosh^2(2x) = \frac{1}{2}(1 + \cosh(2x))$ |

## 11.1 Derivative of Hyperbolic Functions

**Example 11.1.** Differentiate $y = \sinh(x)$.
By the definition, $\sinh(x) = \frac{e^x - e^{-x}}{2}$ and therefore,

$$\frac{d}{dx}\sinh(x) = \frac{d}{dx}\left(\frac{e^x - e^{-x}}{2}\right) =$$

**Theorem 11.1** (Derivatives of Hyperbolic Functions)**.** Let $u = g(x)$ be a differentiable functions.

| | |
|---|---|
| $\frac{d}{dx}\sinh(u) = \cosh(u)\frac{du}{dx}$ | $\frac{d}{dx}\cosh(u) = \sinh(u) = \frac{du}{dx}$ |
| $\frac{d}{dx}\tanh(u) = \text{sech}^2\frac{du}{dx}$ | $\frac{d}{dx}\coth(u) = -csch^2(u)\frac{du}{dx}$ |
| $\frac{d}{dx}\text{sech}(u) = \text{sech}(u)\tanh(u)\frac{du}{dx}$ | $\frac{d}{dx}\text{csch}(u) = -\text{csch}(u)\coth(u)\frac{du}{dx}$ |

**Example 11.2.** Differentiate (a) $y = \sinh(\sqrt{2x+1})$ (b) $y = \coth(x^3)$.

## 11.2 Inverse Hyperbolic Functions

**Theorem 11.2.** Logarithmic Identities for Inverse Hyperbolic Functions.

$\sinh^{-1}(x) = \ln\left(x + \sqrt{x^2+1}\right)$ $\quad$ $\cosh^{-1}(x) = \ln\left(x + \sqrt{x^2-1}\right), \quad x \geq 1$

$\tanh^{-1}(x) = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right), \quad |x| < 1$ $\quad$ $\coth^{-1}(x) = \frac{1}{2}\ln\left(\frac{x+1}{x-1}\right), \quad |x| > 1$

$\operatorname{sech}^{-1}(x) = \ln\left(\frac{1+\sqrt{1-x^2}}{x}\right), \ 0 < x \leq 1$ $\quad$ $\operatorname{csch}^{-1}(x) = \ln\left(\frac{1}{x} + \frac{\sqrt{1+x^2}}{|x|}\right), \quad x \neq 0$

The derivatives of the inverse hyperbolic functions given next can be computed by using the *implicit differentiation* or by using the identities from the above theorem.

**Derivatives of Inverse Hyperbolic Functions** Let $u = g(x)$ be a differentiable function.

$\frac{d}{dx}\sinh^{-1}(u) = \frac{1}{\sqrt{u^2+1}}\frac{du}{dx}$ $\quad$ $\frac{d}{dx}\cosh^{-1}(u) = \frac{-1}{\sqrt{u^2-1}}\frac{du}{dx}, \quad u > 1$

$\frac{d}{dx}\tanh^{-1}(u) = \frac{1}{1-u^2}\frac{du}{dx}, \quad |u| < 1$ $\quad$ $\frac{d}{dx}\coth^{-1}(u) = \frac{1}{1-u^2}\frac{du}{dx}, \quad |u| > 1$

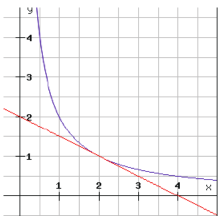
$\frac{d}{dx}\operatorname{sech}^{-1}(u) = \frac{-1}{u\sqrt{1-u^2}}\frac{du}{dx}, \ 0 < u < 1$ $\quad$ $\frac{d}{dx}\operatorname{csch}^{-1}(u) = \frac{-1}{|u|\sqrt{1+u^2}}\frac{du}{dx}, \quad u \neq 0.$

**Example 11.3.** Differentiate $y = \sinh^{-1}(e^x + \sin(2x))$.

**Example 11.4** (Exercise)**.** Differentiate $y = \left(\tanh^{-1}(x^x)\right)^{-1/3}$.

## 12 Linearization and Differentials

Recall that an equation of the tangent line of $y = f(x)$ at the point $(a, f(a))$ is in the form of $y - f(a) = f'(a)(x - a)$ *or* $y = f'(a)(x - a) + f(a)$.




This tangent line equation can be viewed as a function of the variable $x$: $\boxed{L(x) = f'(a)(x - a) + f(a)}$ and this function will be given a special name.

**Definition 12.1** (Linearization)**.** Suppose a function $y = f(x)$ is differentiable at a number $a$. Then a **linearization of** $f$ **at** $a$ is the function given by

$$L(x) = f'(a)(x - a) + f(a).$$

For any $x$ with the value close to $a$, the function value $f(x)$ can be approximated by $L(x)$, i.e.,

$$f(x) \approx L(x)$$

and this approximation is called **a local linear approximation of** $f$ **at** $a$.

**Example 12.1.** Find a linearization of $f(x) = \sin(2x + x^2)$ at $a = 0$.

**Example 12.2.** (Linearization and Approximation)
(a) Find a linearization of $f(x) = \sqrt{x+1}$ at $a = 3$.

(b) Use the linearization in (a) to approximate $\sqrt{3.95}$ and $\sqrt{4.01}$. [Ans: 1.9875, 2.0025]

## Differentials

The fundamental idea of a linearization of a function is closely related to the terminology *differentials*. Let $y = f(x)$ be a differentiable function at some interval containing $a$ and $a + \triangle x$. When we change the value of $x$ from $a$ to $a + \triangle x$, then the different is

$$\triangle x = (a + \triangle x) - a$$

and the function value will be changed from $f(a)$ to $f(a + \triangle x)$, which can be denoted by

$$\triangle y := f(a + \triangle x) - f(a).$$

We will introduce new notations $dx$ and $dy$ where $dx := \triangle x$ and $dy$ will be defined based on the definition of derivative: $f'(a) = \lim\limits_{\triangle x \to 0} \frac{\triangle y}{\triangle x}$. For small $\triangle x$, we will have

$$f'(a) \approx \frac{\triangle y}{\triangle x} \qquad \Rightarrow \qquad \triangle y \approx f'(a)\triangle x := dy.$$

That is, $\boxed{dx := \triangle x, \qquad dy := f'(a)\triangle x = f'(a)dx.}$
These notations for **differentials** $dx$ and $dy$ will be defined formally as follows.

**Definition 12.2.** The *differential of the independent variable* $x$ is the nonzero $\triangle x$ by $dx$, i.e.

$$dx = \triangle x,$$

which is the difference of the variable $x$.
If $f$ is a differentiable function at $x$, then the *differential of the dependence variable* $y$ is denoted by $dy$, i.e.,

$$dy = f'(x)\triangle x = f'(x)dx.$$

We can compare the differentials defined above with the linearization as follow. For $x = a + \triangle x$, the linearlization of $f(x)$ at $a$: $L(x) = f(a) + f'(a)(x - a)$ can be written as

$$L(a+\triangle x) = f(a) + \underbrace{f'(a)\triangle x}_{dy} \qquad \Rightarrow \qquad L(a+\triangle x) = f(a) + dy \qquad \Rightarrow \qquad dy = L(a+\triangle x) - f(a).$$

Since $\triangle y := f(a + \triangle x) - f(a)$ and $f(a + \triangle x) \approx L(a + \triangle x)$., then we can use $\boxed{\triangle y \approx dy}$. That is, we can view the differential $dy$ as an **approximate of the difference** (or error) in $f(x)$ as the value in $x$ is changed by $\triangle x$.

**Example 12.3.** Find $dy$ for $y = x^2 \cos(3x)$.

$$dy = \left(\frac{dy}{dx}\right) \cdot dx =$$

**Example 12.4.** Find $dy$ for $y = x^2 + e^{\sin(2x)}$.

**Example 12.5.** (Differentials)
(a) Find $\triangle y$ and $dy$ for $f(x) = 5x^2 + 4x + 1$.
(b) Compare the values of $\triangle y$ and $dy$ for $x = 6$, $\triangle x = 0.02$
[Ans: (a) $\triangle y = 10x\triangle x + 4\triangle x + 5(\triangle x)^2$, $dy = (10x + 4)dx$; (b)$\triangle y = 1.282$, $dy = 1.28$ ]

**Example 12.6.** (Approximation by Differentials) A side of a cube is measured to be 30 cm with a possible error of $\pm 0.02$ cm. What is the approximate maximum possible error in the volume of the cube?

Solution: The volume of a cube is $V(x) = x^3$, where $x$ is the length of each side. Suppose $\triangle x$ is used for the error in measurement the length for each side. Then the error in obtaining the volume is

$$\triangle V = (x + \triangle x)^3 - x^3.$$

We can approximate this error for the volume by using

$$dV \approx \triangle V$$

where $dV$ is the **differential** of $V(x) = x^3$. That is, $dV = V'(30)dx$. Since $V'(x) = 3x^2$ and $dx = \triangle x = \pm 0.02$,

$$\triangle V \approx dV = V'(30)dx = (3)(30)^2(\pm 0.02) = \pm 54 \text{ cm}^3.$$

Notice that, to obtain a reasonable error measurement, it should be compared with the volume at $x = 30$, i.e.$V(30) = 30^3 = 27,000$ and hence the relative error should be used and given by $\frac{dV}{V} = \frac{\pm 54}{27{,}000} = \pm \frac{1}{500}$. ■