Solution: Assignment 4 (Part I)

1. Let a, b, and c be real numbers. Show that if $a \geq b$ and $c < 0$, then $a^2 + b^2 + 5a - 5b - 4c > 0$.

   Answer:

   (i) Since $a^2 \geq 0$, $b^2 \geq 0$, then $a^2 + b^2 \geq 0$.

   (ii) Since $a \geq b$, then $a - b \geq 0$ and $5(a - b) \geq 0$ or $5a - 5b \geq 0$.

   (iii) Since $c < 0$, then $-4c > 0$.

   From (i) and (ii),
   $$a^2 + b^2 + (5b - 5b) \geq 0$$
   and by adding $-4c$ both sides and use (iii), we have
   $$\begin{aligned} a^2 + b^2 + (5b - 5b) - 4c &\geq -4c \\ &> 0. \end{aligned}$$
   That is, $a^2 + b^2 + 5b - 5b - 4c > 0$. ■

2. Let x, y be real numbers. Suppose that $x < 0$ and $y > 3$. Determin whether each of the following inequalities is true or not. Explain your answer.

   (a) $3x - xy > 0$.

   (b) $\frac{3}{x} - \frac{y}{x} + x^2 + y^3 > 18$

   Answer:

   (a) $3x - xy > 0$ is true. First note that, since $y > 3$, then
   $$3 - y < 0 \tag{1}$$
   and by multiplying throughout the inequality with a negative number x ($x < 0$), the inequality sign is changed:
   $$x(3 - y) > x \cdot 0$$
   or
   $$3x - xy > 0.$$
   ■

   (b) $\frac{3}{x} - \frac{y}{x} + x^2 + y^3 > 18$ is true. Notice again that since $y > 3$, then $3 - y < 0$. Since $x < 0$, then $\frac{1}{x} < 0$ and by multiplying $\frac{1}{x}$ throughout the inequality, we have
   $$\frac{1}{x}(3 - y) > \frac{1}{x} \cdot 0 \qquad \text{or} \qquad \frac{3}{x} - \frac{y}{x} > 0.$$

Since $x < 0$, then $\boxed{x^2 > 0}$ and since $y > 3$, then $\boxed{y^3 > 27}$. That is,

$$\begin{aligned} \frac{3}{x} - \frac{y}{x} &> 0 \\ \frac{3}{x} - \frac{y}{x} + x^2 &> x^2 \\ &> 0 \\ \frac{3}{x} - \frac{y}{x} + x^2 + y^3 &> y^3 \\ &> 27 \\ &> 18 \end{aligned}$$

That is, $\frac{3}{x} - \frac{y}{x} + x^2 + y^3 > 18$.

3. Find the solution set for each of following inequalities.

(a)

$$\frac{2x}{1-x} \geq \frac{1-x}{2x}$$

Answer:

$$\begin{aligned} \frac{2x}{1-x} &\geq \frac{1-x}{2x} \\ \frac{2x}{1-x} - \frac{1-x}{2x} &\geq 0 \end{aligned}$$

Notice that

$$\frac{2x}{1-x} - \frac{1-x}{2x} = \frac{(2x)^2 - (1-x)^2}{2x(1-x)} = \frac{4x^2 - (1-2x+x^2)}{2x(1-x)} = \frac{3x^2+2x-1}{2x(1-x)} = \frac{(3x-1)(x+1)}{2x(1-x)}.$$

Therefore the solution set of the given inequality is the same as the one of

$$\frac{(3x-1)(x+1)}{2x(1-x)} \geq 0$$

which can be solved by looking at the intervals constructed from $x = -1, 0, \frac{1}{3}, 1$.

	$x \in (-\infty, -1)$	$x \in (-1, 0)$	$x \in (0, 1/3)$	$x \in (1/3, 1)$	$x \in (1, \infty)$
$(3x-1)$	$-$	$-$	$-$	$+$	$+$
$(x+1)$	$-$	$+$	$+$	$+$	$+$
x	$-$	$-$	$+$	$+$	$+$
$(1-x)$	$+$	$+$	$+$	$+$	$-$
$\frac{(3x-1)(x+1)}{2x(1-x)}$	$-$	$\boxed{+}$	$-$	$\boxed{+}$	$-$

To have non-zero denominator, we must have $x \neq 0$, $x \neq 1$. Since we have "greater than or equal" sign, then the solution set is

$$[-1, 0) \cup [1/3, 1).$$

■

(b)

$$\frac{16x^4 - 81}{6x^2 + x - 12} < 0$$

Answer: Notice that

$$16x^4 - 81 = (4x^2)^2 - 9^2 = (4x^2 - 9)(4x^2 + 9) = (2x - 3)(2x + 3)(4x^2 + 9)$$

and

$$6x^2 + x - 12 = (2x + 3)(3x - 4).$$

That is,

$$\frac{16x^4 - 81}{6x^2 + x - 12} = \frac{(2x - 3)(2x + 3)(4x^2 + 9)}{(2x + 3)(3x - 4)} = \frac{(2x - 3)(4x^2 + 9)}{(3x - 4)} < 0$$

with $(2x + 3) \neq 0$ or $x \neq -\frac{3}{2}$. Since $4x^2 + 9 > 0$ for all $x \in \mathbb{R}$, then we only need to look at the sign of $\frac{(2x-3)}{(3x-4)} < 0$. Consider the intervals divided by $x = 3/2$, $x = 4/3$.

	$x \in (-\infty, 3/2)$	$x \in (3/2, 4/3)$	$x \in (4/3, \infty)$
$(2x - 3)$	$-$	$+$	$+$
$(3x - 4)$	$-$	$-$	$+$
$\frac{(2x-3)}{(3x-4)}$	$+$	$\boxed{-}$	$+$

Therefore, the solution set is $(3/2, 4/3)$. ■

(c)

$$\frac{x^4 - 2x^2 - 8}{2x + 1} \geq 0$$

Answer: Notice that

$$x^4 - 2x^2 - 8 = (x^2 - 4)(x^2 + 2) = (x - 2)(x + 2)(x^2 + 2).$$

and $\frac{x^4-2x^2-8}{2x+1} \geq 0$ is equivalent to

$$\frac{(x - 2)(x + 2)(x^2 + 2)}{2x + 1} \geq 0.$$

Since $x^2 + 2 > 0$ for all $x \in \mathbb{R}$, we will consider $\frac{(x-2)(x+2)}{2x+1} \geq 0$. The rational function $\frac{(x-2)(x+2)}{2x+1}$ has zeros at $x = -2, 2$ and an undefined point at $x = -1/2$.

	$x \in (-\infty, -2)$	$x \in (-2, -1/2)$	$x \in (-1/2, 2)$	$x \in (2, \infty)$
$x + 2$	$-$	$+$	$+$	$+$
$2x + 1$	$-$	$-$	$+$	$+$
$x - 2$	$-$	$-$	$-$	$+$
$\frac{(x-2)(x+2)}{2x+1}$	$-$	$\boxed{+}$	$-$	$\boxed{+}$

To have non-zero denominator, we must have $x \neq -1/2$. Since we have "greater than or equal" sign, then the solution set is $[-2, -1/2) \cup [2, \infty)$. ■

(d)

$$\frac{x^4 + x^3 + 2x^2 + 2x}{x - 1} \leq 0$$

Answer: Notice that

$$x^4 + x^3 + 2x^2 + 2x = x^2(x^2 + 2) + x(x^2 + 2) = (x^2 + x)(x^2 + 2) = x(x + 1)(x^2 + 2).$$

That is, $\frac{x^4+x^3+2x^2+2x}{x-1} \leq 0$ is equivalent to

$$\frac{x(x + 1)(x^2 + 2)}{x - 1} \leq 0.$$

Note that $x^2 + 2 > 0$ for any real number x so we need to only consider $\frac{x(x+1)}{x-1} \leq 0$. To obtain the subintervals, consider $x = -1, 0, 1$

	$x \in (-\infty, -1)$	$x \in (-1, 0)$	$x \in (0, 1)$	$x \in (1, \infty)$
$x + 1$	$-$	$+$	$+$	$+$
x	$-$	$-$	$+$	$+$
$x - 1$	$-$	$-$	$-$	$+$
$\frac{x(x+1)}{x-1}$	$\boxed{-}$	$+$	$\boxed{-}$	$+$

To have non-zero denominator, we must have $x \neq 1$. Since we have "less than or equal" sign, then the solution set is $(-\infty, -1] \cup [0, 1)$. ■

(e)

$$\left(\frac{x}{x - 3} - 2\right)\left(\frac{e^x}{\cos(x) + 2}\right) \geq 0$$

Answer: Notice that $e^x > 0$ for all real numbers x and

$$\begin{aligned} -1 &< \cos(x) < 1 \\ -1 + 2 &< \cos(x) + 2 < 1 + 2 \\ 1 &< \cos(x) + 2 < 3 \end{aligned}$$

That is, $\cos(x) + 2 > 0$ and $\left(\frac{e^x}{\cos(x)+2}\right) > 0$ for any real number x. So, the solution set of the given inequality can be found from solving $\left(\frac{x}{x-3} - 2\right) \geq 0$. Note that

$$\frac{x}{x - 3} - 2 = \frac{x - 2x + 6}{x - 3} = \frac{-x + 6}{x - 3}.$$

That is, we obtain the solution set by solving $\frac{-x+6}{x-3} \geq 0$ or $\frac{x-6}{x-3} \leq 0$. Consider the intervals divided by $x = 3$, $x = 6$.

	$x \in (-\infty, 3)$	$x \in (3, 6)$	$x \in (6, \infty)$
$x - 3$	$-$	$+$	$+$
$x - 6$	$-$	$-$	$+$
$\frac{x-6}{x-3}$	$+$	$\boxed{-}$	$+$

To have non-zero denominator, we must have $x \neq 3$. Since we have "less than or equal" sign, then the solution set is $(3, 6]$. ■

4. (Optional) Find the solution set for each of the following inequities.

   (a) $\frac{x-1}{x+1} \leq 0$

   (b) $\frac{-x^2-x}{2-x} \geq 0$

   (c) $\frac{2x}{x-2} - \frac{3x}{x-4} \leq 1$

   (d) $\frac{(2x^2+4x+7)(x-1)}{x^3-5x^2+x-5} \leq 0$

   (e) $\frac{x^2+8}{x^2+x+6} \geq 0$

   (f) $\frac{x^2-2x-4}{x^2-x-6} \geq 0$

   (g) $\frac{2x-x^2-3}{x-2x^2-1} \leq 1$

   (h) $\frac{(-x^2+5x-7)(2x+1)}{x^4-1} \geq 0$

   (i) $\frac{x^2-1}{2-3x+x^2} > \frac{1}{x}$

   (j) $\frac{x^4+6x^2}{1-2x} \leq x^2$

   (k) $x^5 + 3x^4 - 23x^3 - 51x^2 + 94x + 120 \geq 0$