Tanatat Bhurivatana
6204640137

2. Assume there is an economy with k states of nature and where the following asset pricing formula holds:

$$\begin{aligned} P_a &= \sum_{s=1}^{k} \pi_s m_s X_{sa} \\ &= E\left[mX_a\right] \end{aligned}$$

Let an individual in this economy have the utility function $\ln(C_0) + E[\delta \ln(C_1)]$, and let C_0^* be her equilibrium consumption at date 0 and C_s^* be her equilibrium consumption at date 1 in state s, $s = 1, ..., k$. Denote the date 0 price of elementary security s as p_s, and derive an expression for it in terms of the individual's equilibrium consumption.

Utility function ; $\ln(C_0) + E[\delta \ln(C_1)]$

① $P_s = \pi_s M_s \, 1$

② $m_s = \dfrac{\delta U'(C_s^*)}{U'(C_0^*)} = \dfrac{\delta\left(\frac{1}{C_s^*}\right)}{\left(\frac{1}{C_0^*}\right)} = \dfrac{\delta C_0^*}{C_s^*}$

$P_s = \pi_s \dfrac{\delta C_0^*}{C_s^*}$

3. Consider the one-period consumption - portfolio choice problem. The individual's first order conditions leads to the general relationship

$$1 = E\left[m_{01} R_s\right] \quad = \pi_s m_{01} R_s$$

where m_{01} is the stochastic discount factor between dates 0 and 1 and R_s is the one-period stochastic return on any security in which the individual can invest. Let there be a finite number of date 1 states where π_s is the probability of state s. Also assume markets are complete and consider the above relationship for primitive security s, that is, let R_s be the rate of return on primitive (or elementary) security s. The individual's elasticity of intertemporal substitution is defined as

$$\varepsilon^I \equiv \frac{R_s}{C_s/C_0} \frac{d\left(C_s/C_0\right)}{dR_s}$$

where C_0 is the individual's consumption at date 0 and C_s is the individual's consumption at date 1 in state s. If the individual's expected utility is given by

$$U\left(C_0\right) + \delta E\left[U\left(\tilde{C}_1\right)\right]$$

where utility displays constant relative risk aversion, $U(C) = C^\gamma/\gamma$, solve for the elasticity of intertemporal substitution, ε^I.

$$U'(C) = \frac{\gamma C^{\gamma-1}}{\gamma} = C^{\gamma-1}$$

$$m_{01} = \frac{\delta U'(C_s)}{U'(C_0)} = \frac{\delta C_s^{\gamma-1}}{C_0^{\gamma-1}} = \delta\left(\frac{C_s}{C_0}\right)^{\gamma-1}$$

$$1 = E\left[m_{01} R_s\right] \qquad 1 = \underbrace{\pi_s}_{\text{prob}} \delta \left(\frac{C_s}{C_0}\right)^{\gamma-1} R_s \qquad (1)$$

Total diff (1), $\cancel{\pi_s \delta}\left(\left(\frac{C_s}{C_0}\right)^{\gamma-1} dR_s + (R_s)\left[(\gamma-1)\left(\frac{C_s}{C_0}\right)^{\gamma-2} d\left(\frac{C_s}{C_0}\right)\right]\right) = 0$

$$R_s(\gamma-1)\left(\frac{C_s}{C_0}\right)^{\gamma-2} d\left(\frac{C_s}{C_0}\right) = -\left(\frac{C_s}{C_0}\right)^{\gamma-1} dR_s$$

$$R_s(\gamma-1)\left(\frac{C_s}{C_0}\right)^{\gamma-2} \frac{d\left(\frac{C_s}{C_0}\right)}{dR_s} = -\left(\frac{C_s}{C_0}\right)^{\gamma-1}$$

divide $\left(\frac{C_1}{C_0}\right)^{\gamma-2}$ both side

$$R_s(\gamma-1) \frac{d\left(\frac{C_s}{C_0}\right)}{dR_s} = -\frac{C_1}{C_0}^{\gamma-1-\gamma+2}$$

$$R_s(\gamma-1) \frac{d\left(\frac{C_s}{C_0}\right)}{dR_s} = -\frac{C_1}{C_0}$$

$$\frac{R_s}{\left(\frac{C_1}{C_0}\right)} \frac{d\left(\frac{C_s}{C_0}\right)}{dR_s} = -\frac{1}{\gamma-1} \Rightarrow \varepsilon^I$$

4. [16]Consider an economy with $k = 2$ states of nature, a "good" state and a "bad" state. There are two assets, a risk free asset with $R_f = 1.05$, and a second risky asset that pays cashflows

$$X_2 = \begin{bmatrix} 10 \\ 5 \end{bmatrix}$$

The current price of the risky asset is 6.

4.a Solve for the prices of the elementary securities, p_1 and p_2 and the risk-neutral probabilities of the two states.

4.b Suppose that the physical probabilities of the two states are $\pi_1 = \pi_2 = 0.5$. What is the stochastic discount factor for the two states?

4.A

$$X = \begin{array}{cc} x_1 & x_2 \\ \end{array}$$
$$X = \begin{bmatrix} 1 & 10 \\ 1 & 5 \end{bmatrix}$$

$$p_1 = P'X^{-1}e_1$$

$$P = \begin{bmatrix} \frac{1}{R_f} \\ 6 \end{bmatrix} = \begin{bmatrix} \frac{1}{1.05} \\ 6 \end{bmatrix}$$

$$P_1 = P' X^{-1} e_1 = \begin{bmatrix} \frac{1}{1.05} & 6 \end{bmatrix} \begin{bmatrix} 1 & 10 \\ 1 & 5 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{1.05} & 6 \end{bmatrix} \begin{bmatrix} -1 & 2 \\ \frac{1}{5} & \frac{-1}{5} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -\frac{1}{1.05} + \frac{6}{5} & \frac{2}{1.05} - \frac{6}{5} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0.2476 & 0.7048 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$= 0.2476$$

$$P_1 = P' X^{-1} e_2 = \begin{bmatrix} \frac{1}{1.05} & 6 \end{bmatrix} \begin{bmatrix} 1 & 10 \\ 1 & 5 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{1.05} & 6 \end{bmatrix} \begin{bmatrix} -1 & 2 \\ \frac{1}{5} & \frac{-1}{5} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -\frac{1}{1.05} + \frac{6}{5} & \frac{2}{1.05} - \frac{6}{5} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0.2476 & 0.7048 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$= 0.7048$$

Risk-neutral Prob: $\hat{\pi}_s = p_s R_f$

$$\hat{\pi}_1 = p_1 R_f = 0.2476(1.05) = 0.26$$
$$\hat{\pi}_2 = p_2 R_f = 0.7048(1.05) = 0.74$$

4.b $m_s = \frac{p_s}{\pi_s}$

$m_1 = \frac{0.2476}{0.5} = 0.4952$, $m_2 = \frac{0.7048}{0.5} = 1.4096$

6. This question asks you to relate the stochastic discount factor pricing relationship to the CAPM. The CAPM can be expressed as

$$E[R_i] = R_f + \beta_i \gamma$$

where $E[\cdot]$ is the expectation operator, R_i is the realized return on asset i, R_f is the risk-free return, β_i is asset i's beta, and γ is a positive market risk premium. Now, consider a stochastic discount factor of the form

$$m = a + bR_m$$

where a and b are constants and R_m is the realized return on the market portfolio. Also, denote the variance of the return on the market portfolio as σ_m^2.

6.a Derive an expression for γ as a function of a, b, $E[R_m]$, and σ_m^2. (Hint: you may want to start from the equilibrium expression $0 = E[m(R_i - R_f)]$.)

6.b Note that the equation $1 = E[mR_i]$ holds for all assets. Consider the case of the risk-free asset and the case of the market portfolio, and solve for a and b as a function of R_f, $E[R_m]$, and σ_m^2.

6.c Using the formula for a and b in part b, show that $\gamma = E[R_m] - R_f$.

6.a $0 = E[m(R_i - R_f)]$

$0 = E[(a + bR_m)(R_i - R_f)]$

$0 = E[aR_i - aR_f + bR_mR_i - bR_mR_f]$

$0 = a\,E[R_i] - aR_f + b\,E[R_mR_i] - bR_f E[R_m]$

$0 = a[E[R_i] - R_f] + b[E[R_mR_i] - R_f E[R_m]]$

$0 = a[E[R_i] - R_f] + b[E[R_m]E[R_i] + Cov[R_m, R_i] - R_f E[R_m]]$

$0 = a[E[R_i] - R_f] + b[E[R_m][E[R_i] - R_f] + Cov[R_m, R_i]]$

$0 = [E(R_i) - R_f][a + bE[R_m]] + b\,Cov(R_m, R_i)$

$\frac{-b\,Cov(R_m, R_i)}{a + bE[R_m]} = E[R_i] - R_f$ (1)

$\beta_i = \frac{Cov[R_i, R_m]}{\sigma^2_m}$

Multiply (1) with $\frac{\sigma^2_m}{\sigma^2_m}$, $E(R_i) - R_f = \frac{-b\,\sigma^2_m}{a + bE(R_m)} \cdot \frac{Cov[R_m, R_i]}{\sigma^2_m}$

$= -\beta_i \frac{b\,\sigma^2_m}{a + b\,E(R_m)}$

CAPM: $E[R_i - R_f] = \beta_i \gamma$

$\gamma = \frac{-b\,\sigma^2_m}{a + b\,E(R_m)}$

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6.b For Risk-free asset :

$$1 = E[m R_f]$$
$$\frac{1}{R_f} = E[m]$$
$$\frac{1}{R_f} = E[a + bR_m] \quad (1)$$

Market Port

$$1 = E[m R_m]$$
$$1 = E[(a+bR_m)R_m]$$
$$1 = E[aR_m + bR_m^2]$$
$$1 = aE[R_m] + bE(R_m^2)$$

Note, $\sigma_m^2 = E[R_m^2] - E[R_m]^2$
$\sigma^2_m + E[R_m]^2 = E[R_m^2]$

$$1 = aE[R_m] + b[\sigma^2_m + E[R_m]^2] \quad (2)$$

Rearrange (1) : $a = \frac{1}{R_f} - bE[R_m] \quad (3)$

$\downarrow$ sub. to (2)

$$1 = \left[\frac{1}{R_f} - bE[R_m]\right]E[R_m] + b[\sigma^2_m + E(R_m)^2]$$
$$1 = \frac{E[R_m]}{R_f} - \cancel{bE[R_m]^2} + b[\sigma^2_m + \cancel{E(R_m)^2}]$$
$$1 = \frac{E(R_m)}{R_f} + b\sigma^2_m$$
$$\frac{1 - \frac{E(R_m)}{R_f}}{\sigma^2_m} = b$$
$$\frac{R_f - E(R_m)}{\sigma^2_m R_f} = b$$

Sub in (3)

$$a = \frac{1}{R_f} - \left(\frac{R_f - E(R_m)}{\sigma^2_m R_f}\right)E(R_m)$$
$$= \frac{1}{R_f} - \left(\frac{E(R_m)\cdot R_f - E(R_m)^2}{\sigma^2_m R_f}\right)$$
$$= \frac{\sigma^2_m - E(R_m)R_f + E(R_m)^2}{\sigma^2_m R_f}$$

(From 6.a)

6.c
$$\gamma = \frac{-b\sigma^2_m}{a + bE(R_m)} = \frac{-\left(\frac{R_f - E(R_m)}{\sigma^2_m R_f}\right)\cdot\sigma^2_m}{\frac{\sigma^2_m - E(R_m)R_f + E(R_m)^2}{\sigma^2_m R_f} + \left(\frac{R_f - E(R_m)}{\sigma^2_m R_f}\right)E(R_m)}$$

$$\gamma = \frac{\frac{-R_f + E(R_m)}{\cancel{R_f}}}{\frac{\cancel{\sigma^2_m}}{\cancel{\sigma^2_m}\cancel{R_f}}} = -R_f + E(R_m) = E(R_m) - R_f$$